Greedy conjecture for strings of length 4

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Part I

Shortest common superstring: overview

Problem statement

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- Practical applications: data storage, data compression, genome assembly.

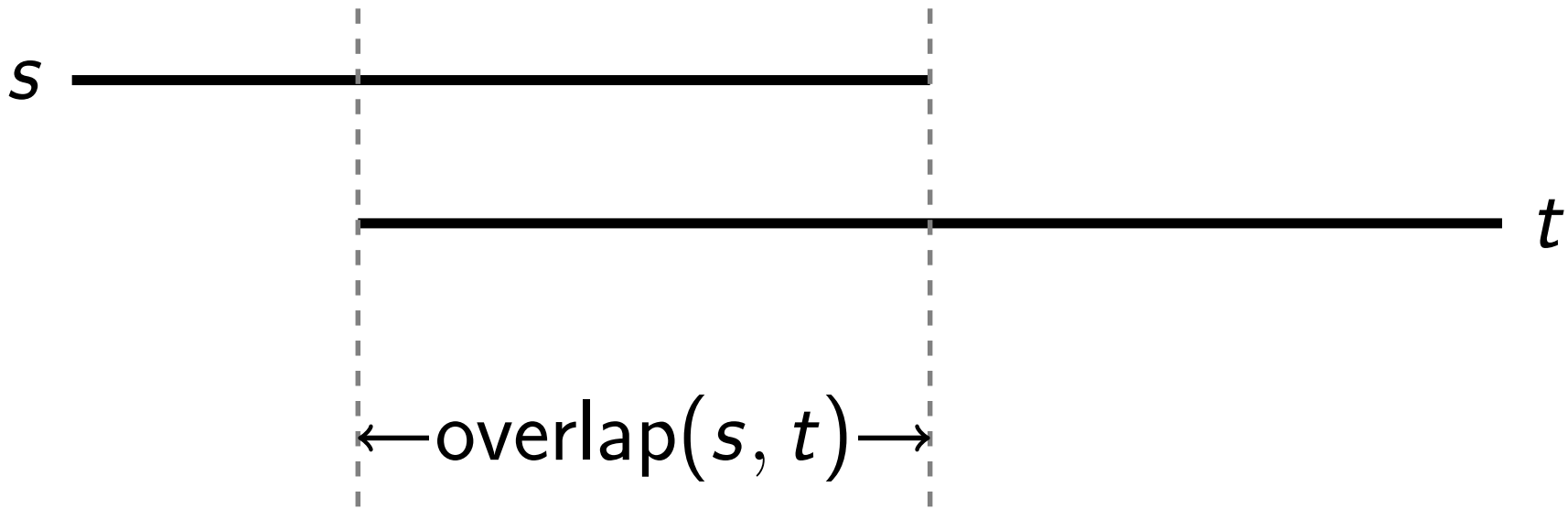
Known results

- Exact: 2^n
  - Dynamic programming, exp space [Bellman, 1960; Held, Karp, 1962]
  - Inclusion-exclusion, poly space [Kohn, Gottlieb, Kohn, 1977; Karp, 1982; Bax, 1993]
  - $2^{n-\Omega(\sqrt{n/\log n})}$ [Björklund, 2012]
- Approximation ratio: $2\frac{11}{23}$ [Mucha, 2013]
- Inapproximability ratio: $1\frac{1}{332}$ [Karpinski and Schmied, 2013]

Known results for special cases

- r-SCS: all input strings have length r.
- 2-SCS $\in$ P, while for $r \geq 3$, r-SCS is NP-hard [Gallant et al., 1980].
- SCS over $\{0, 1\}$ is NP-hard [Gallant et al., 1980].

Overlap



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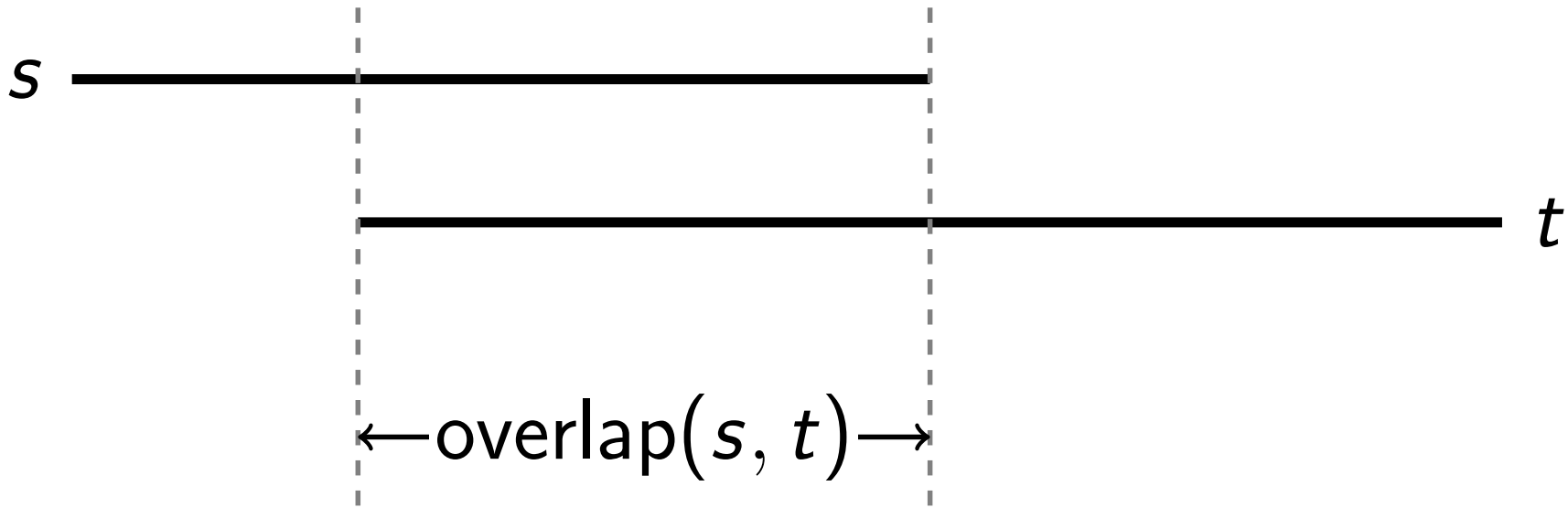

$$|\,\mathrm{overlap}(\mathrm{DAB}, \mathrm{ABE})| = 2$$
$$|\,\mathrm{overlap}(\mathrm{DAB}, \mathrm{ADE})| = 0$$

SCS: permutation problem

ABE DFA DAB ECA CBD ACB

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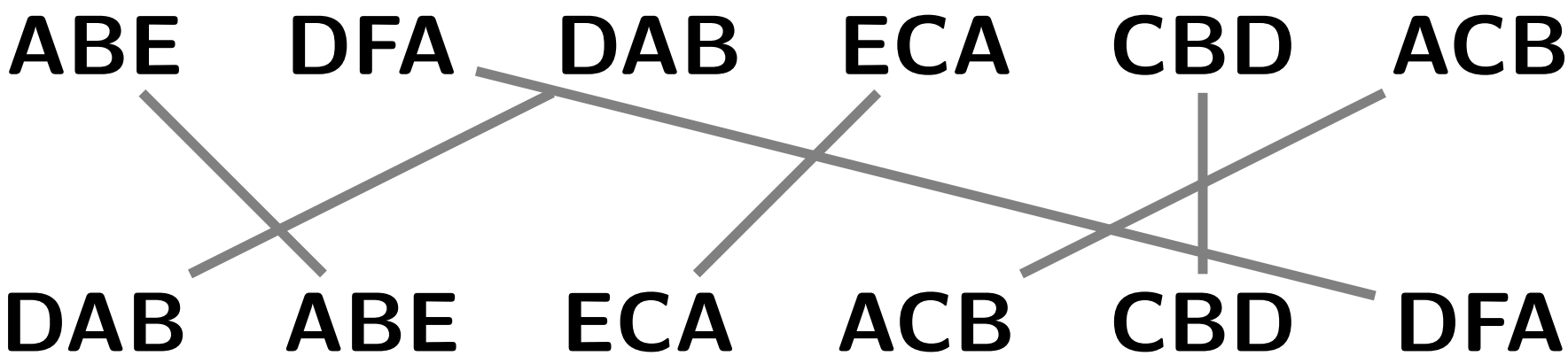

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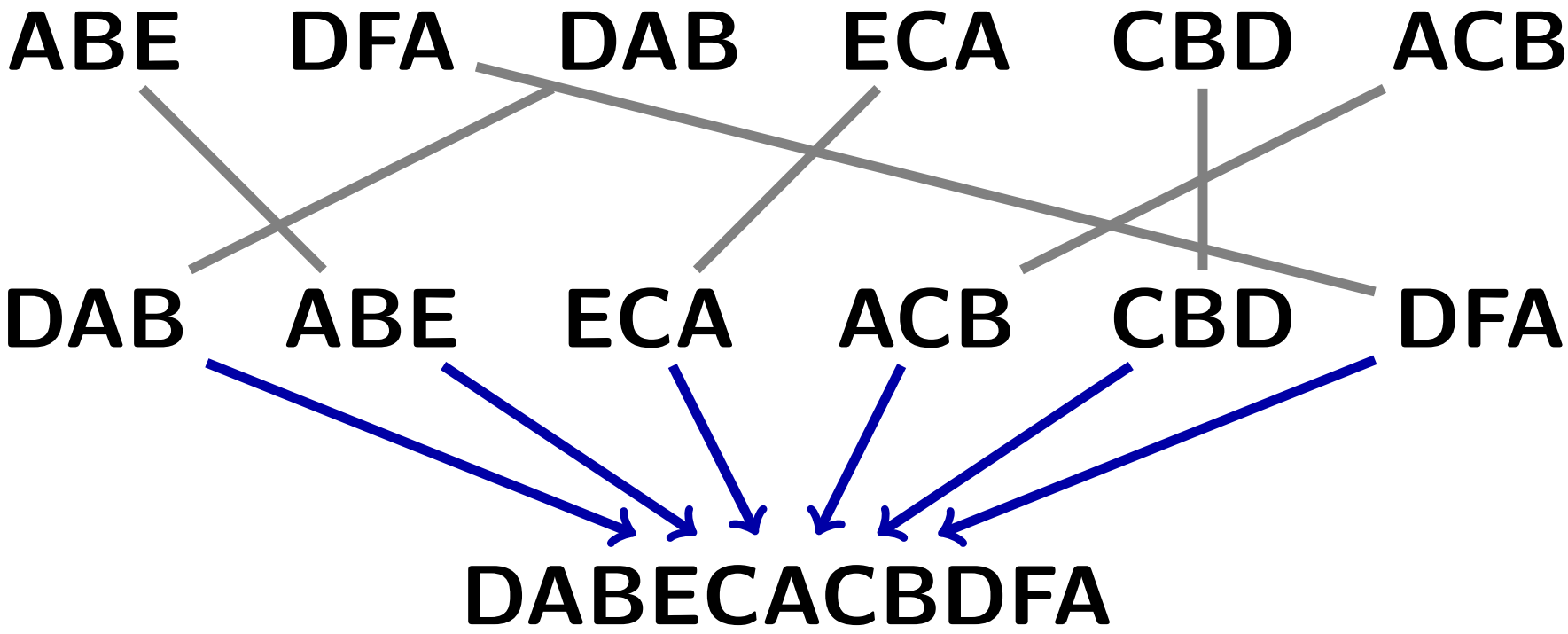

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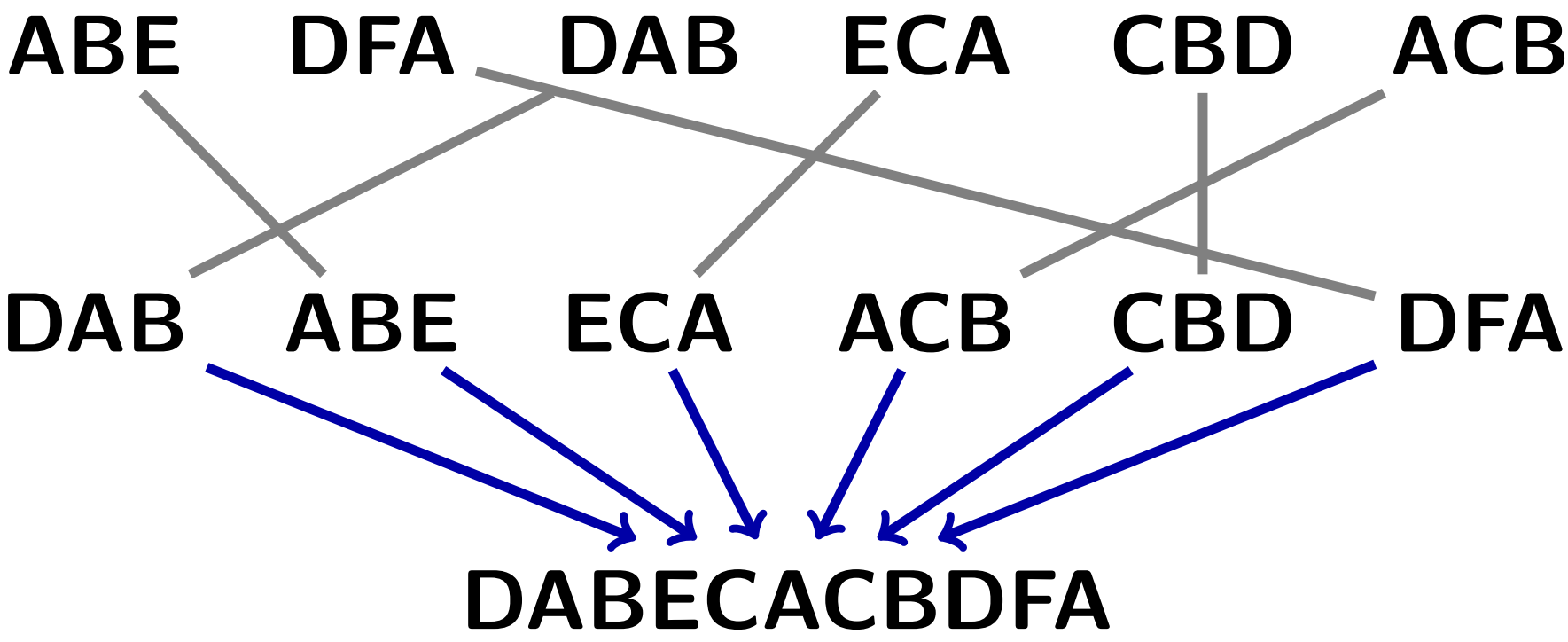

Length of superstring is $\sum_i |s_i| - c^{opt}$

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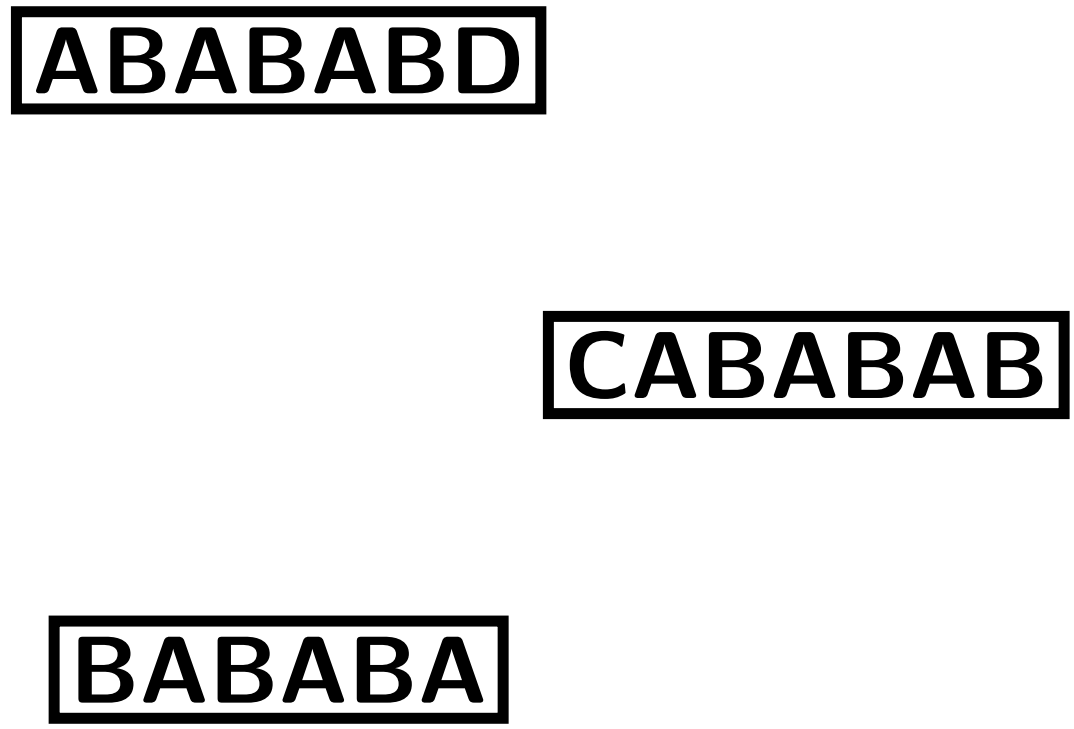

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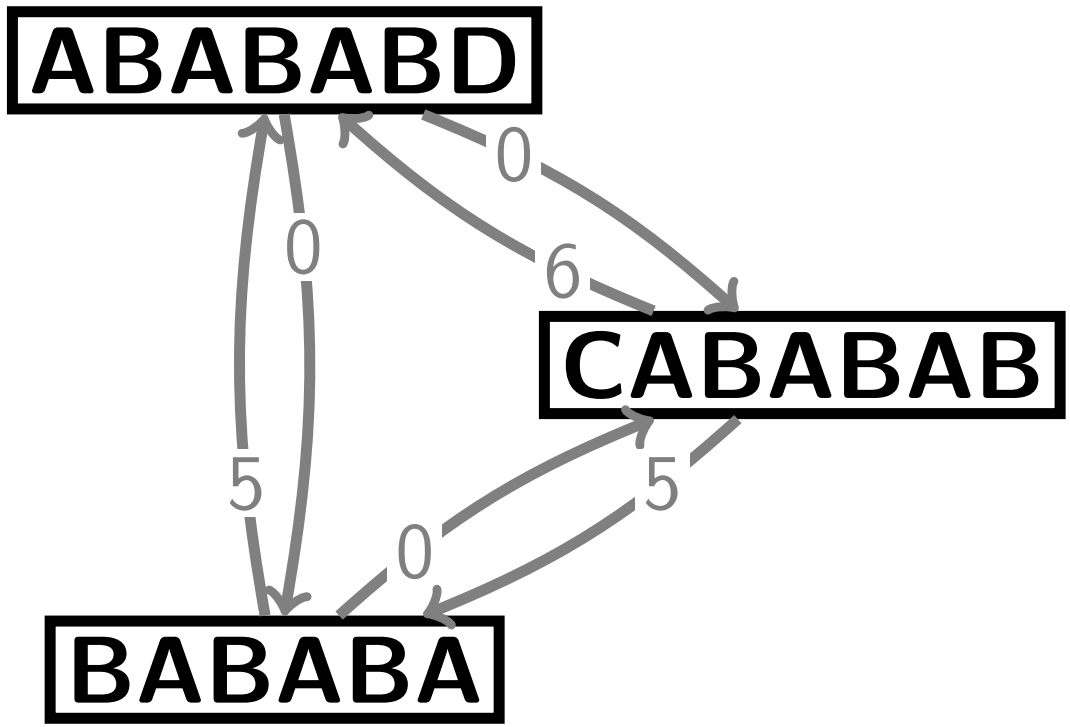

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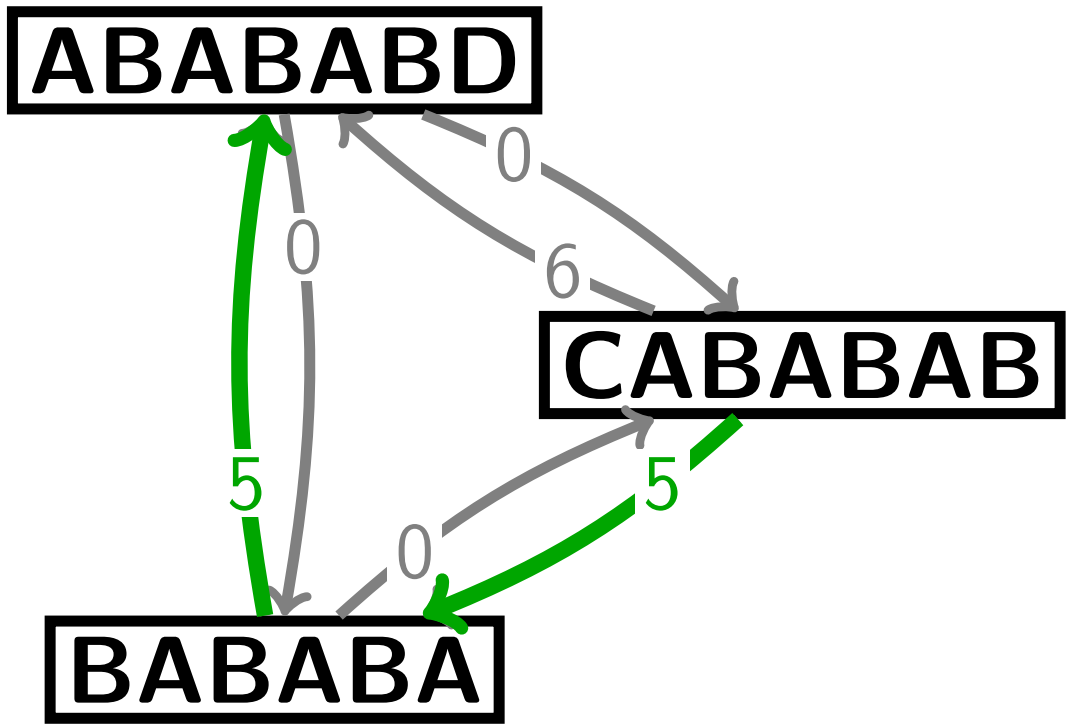

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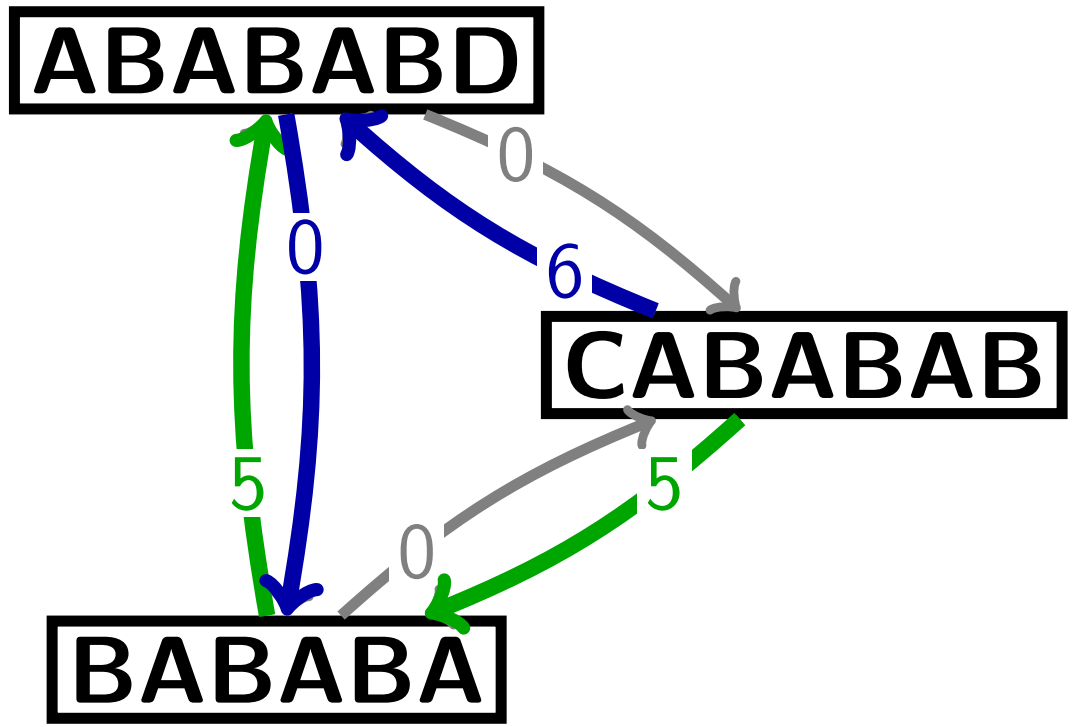

OPT: CABABABABD (10), $c^{opt} = 10$

GREEDY: CABABABDBABABA (14), $c^{gr} = 6$

Greedy conjecture

A superstring resulting from such a greedy process is at most two times longer than an optimal superstring. [Gallant, 1982; Turner, 1989]

Greedy conjecture in terms of compression

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- It is 2-approximation of compression [Tarhio and Ukkonen, 1986]

Part II

Our results

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- This approach does not work for 4-strings
- Thus, to prove the greedy conjecture for 4-strings, we consider overlaps of different length separately

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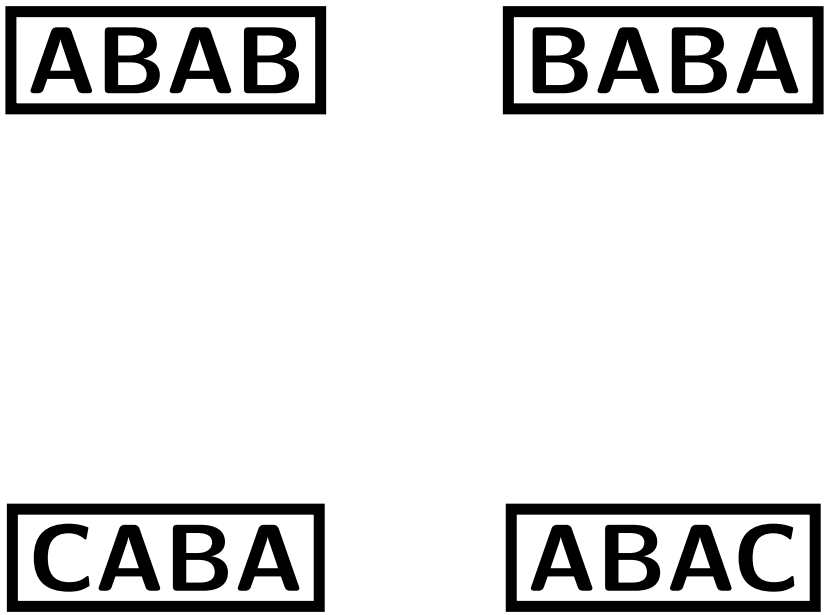
- It is enough to show that $2\#_3^{\text{opt}} \leq 3\#_3^{\text{gr}} + 2\#_2^{\text{gr}} + \#_1^{\text{gr}}$

Analysis of greedy algorithm



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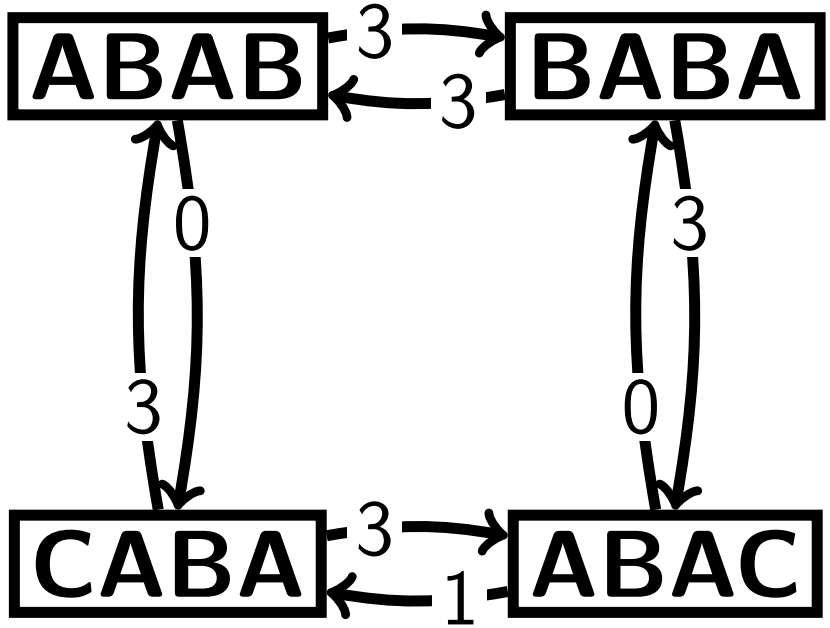

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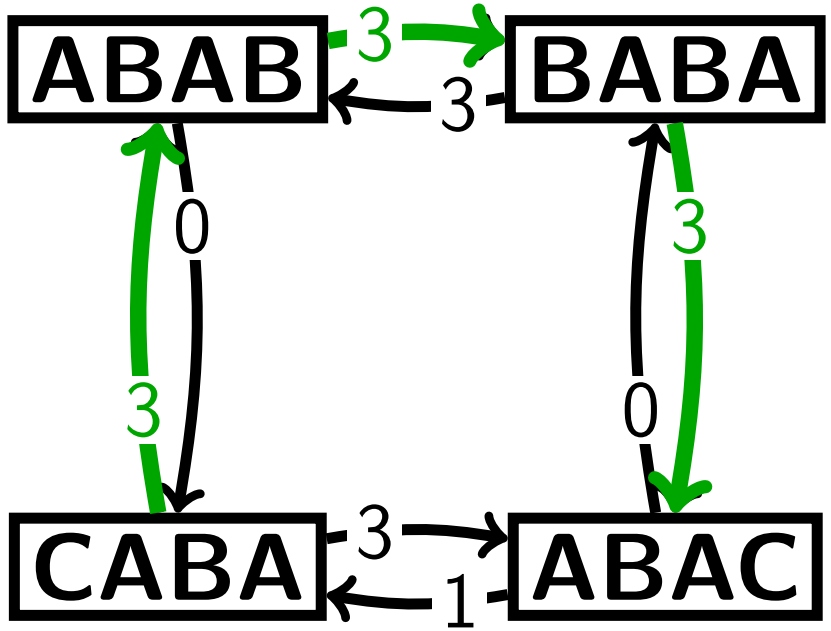

OPT: CABABAC (7), $c^{opt} = 9$

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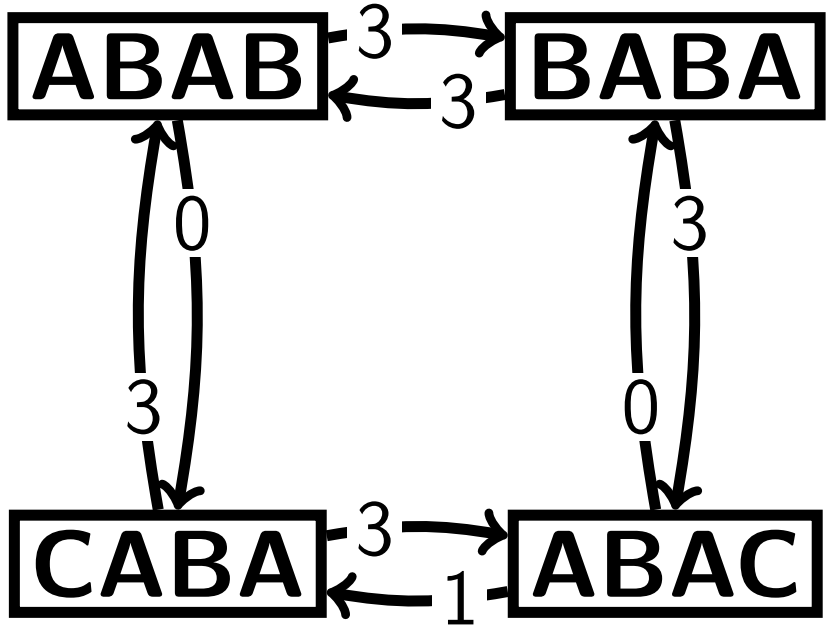

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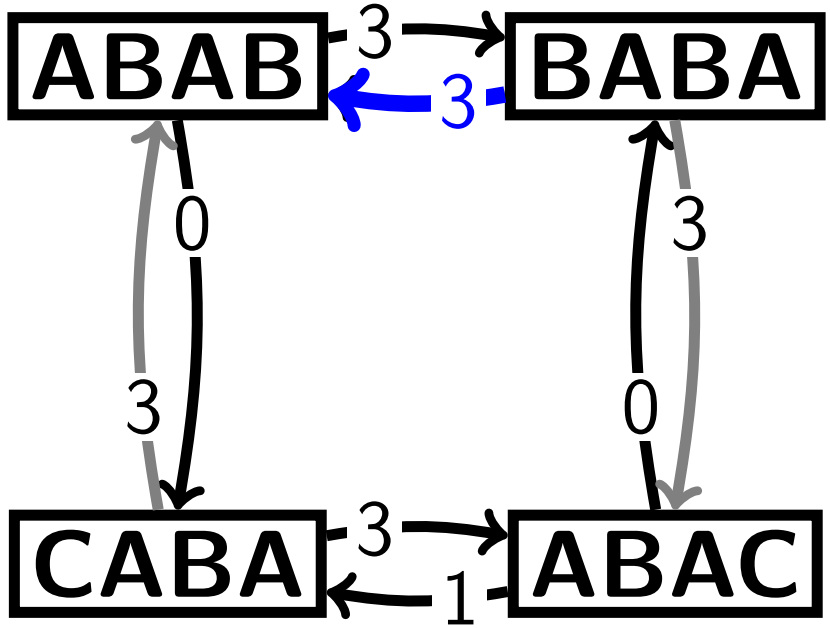

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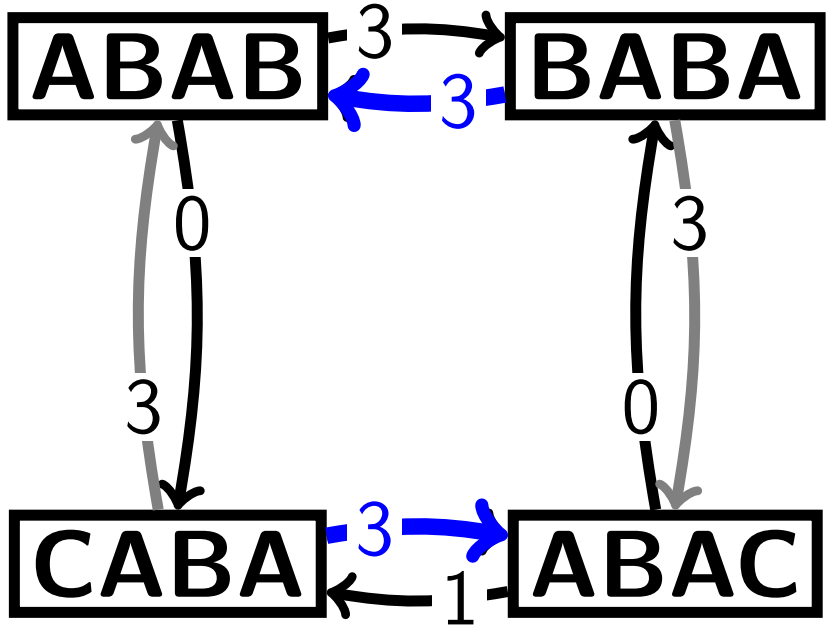

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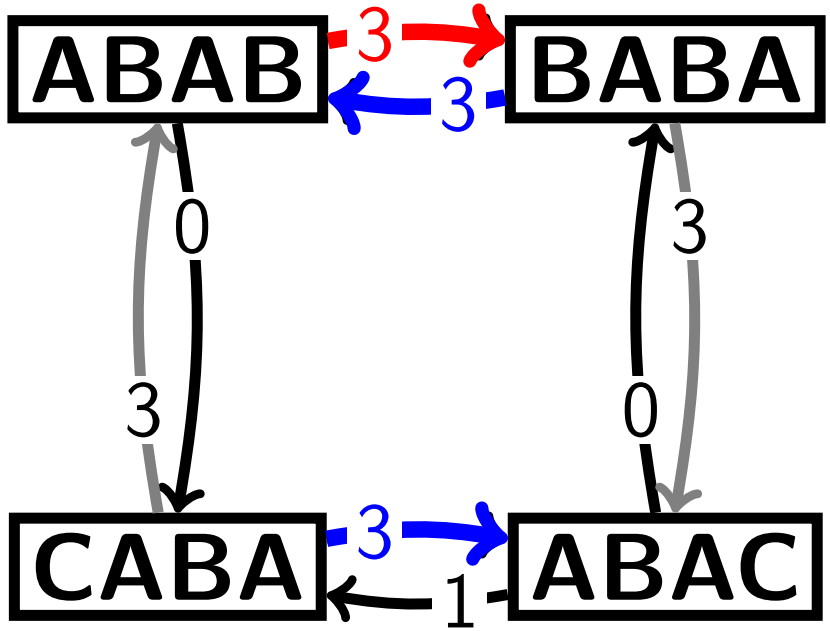

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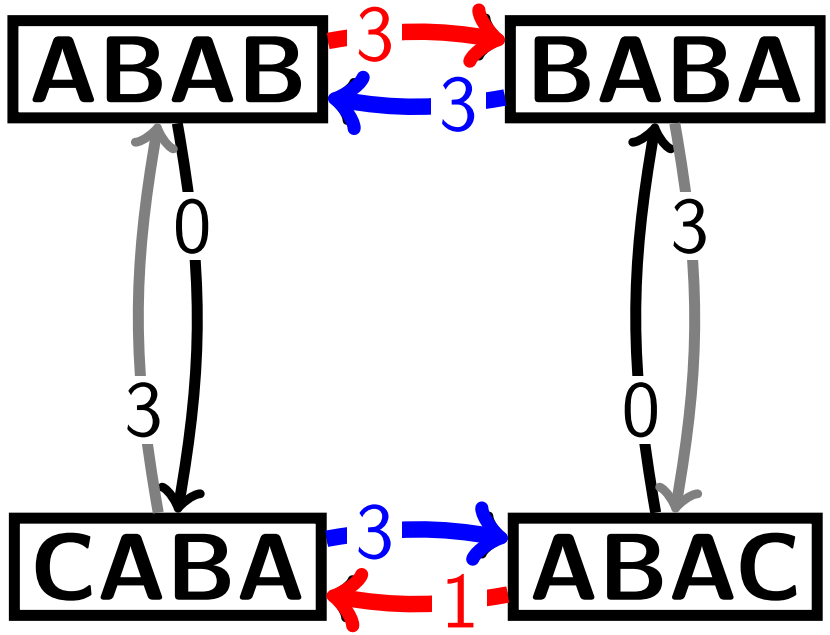

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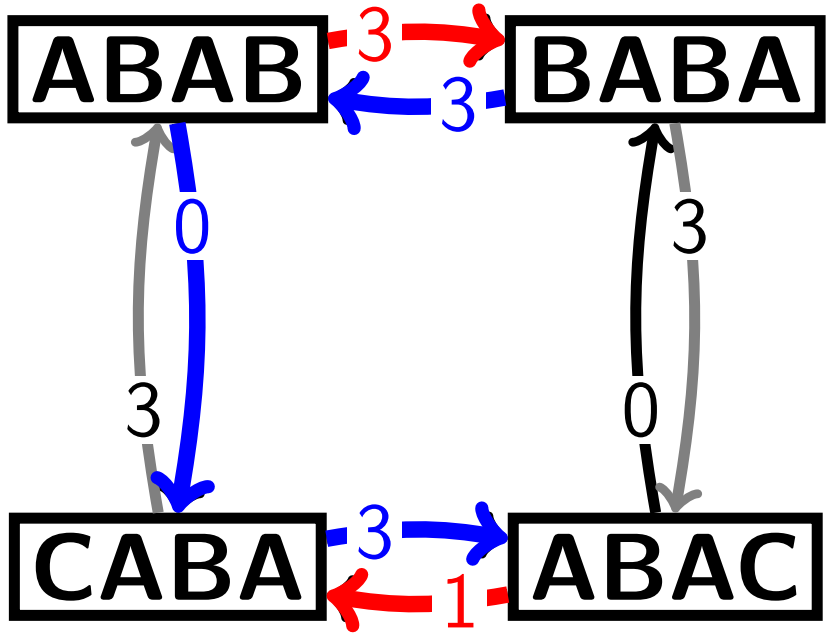

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Observations

- The greedy algorithm finds an optimum cycle cover in the overlap graph [Blum et al., 1994]

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- The only case when the algorithm cannot take an edge is when it forms a cycle. If at each iteration the algorithm takes an edge of maximum weight then the resulting superstring is optimal.

Proof for 4-strings (1)

- Recall that it is enough to show that $2\#_3^{\text{opt}} \leq 3\#_3^{\text{gr}} + 2\#_2^{\text{gr}} + \#_1^{\text{gr}}$

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- Bad situation is when we have a cycle formed by overlaps of length 3
- In case of such cycles with length at least 3 we have a good upper bound for its number: $0.5\#_3^{\text{gr}}$ (due to fact that cycle without one edge contains at least 2 overlaps)

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Thank you for your attention!