On the DCJ Median Problem

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Evolution of Genomes

- Point mutations
  - Base pair substitutions
  - Base pair insertions and deletions

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- Large-scale evolutionary events
  - Content-modifying events
    - Segmental duplications
    - Insertions and deletions
  - Rearrangements
    - Inversions
    - Translocations
    - Chromosomal fissions and fusions

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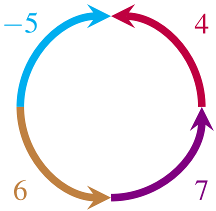

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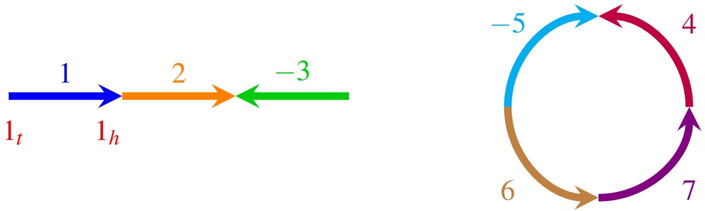

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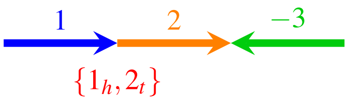

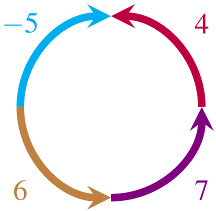

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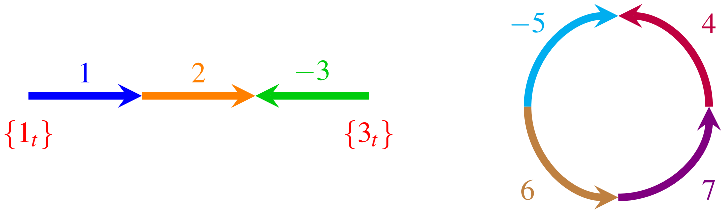

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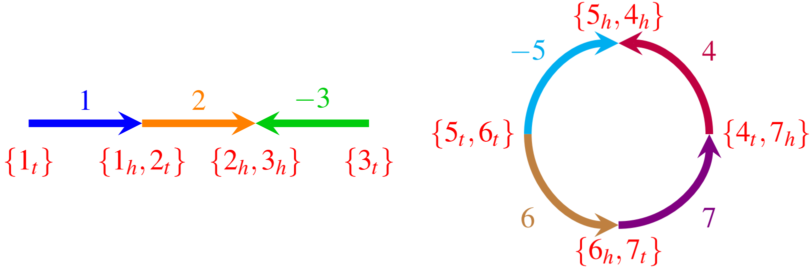

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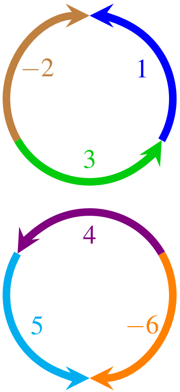

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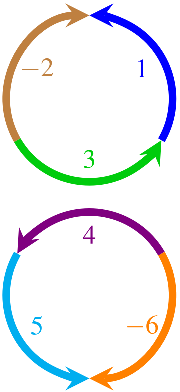

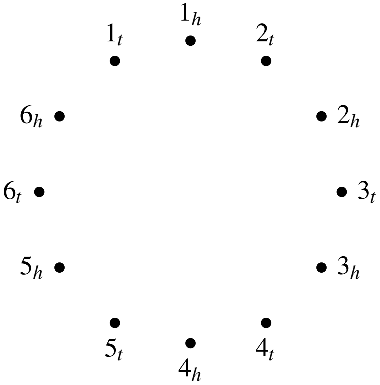

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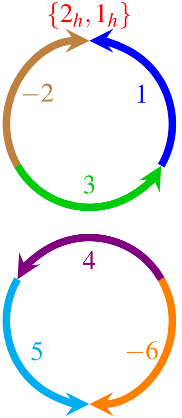

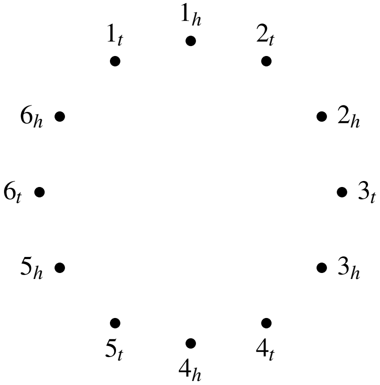

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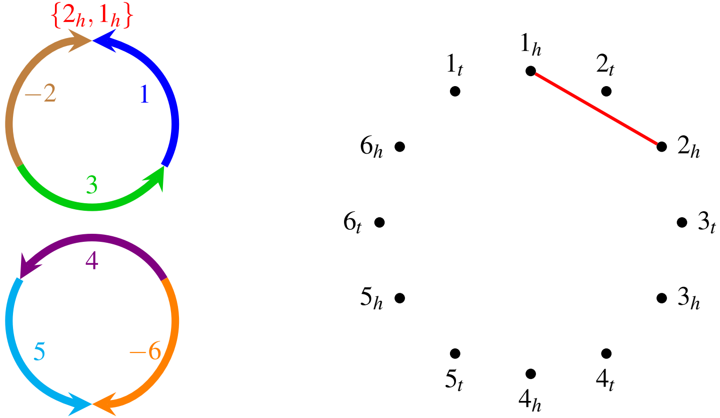

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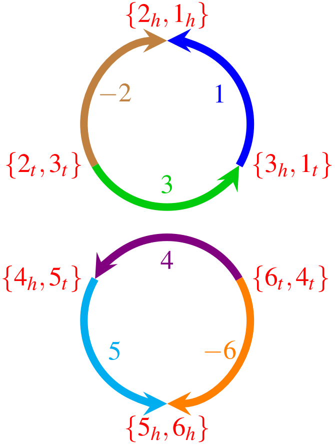

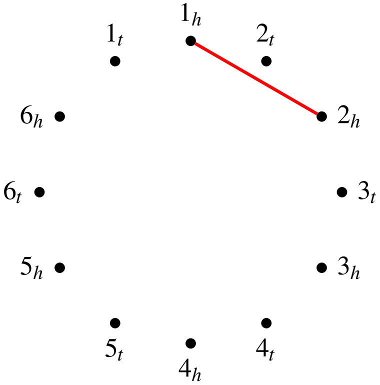

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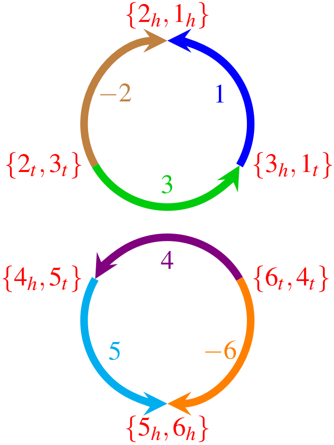

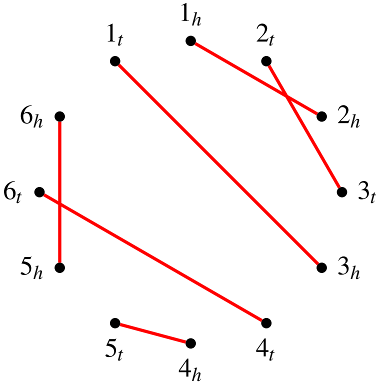

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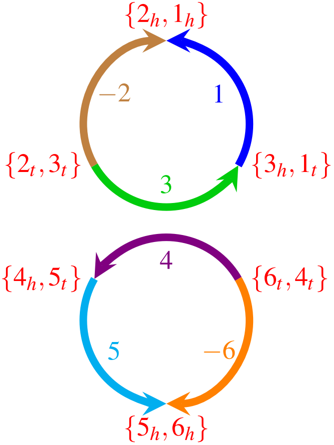

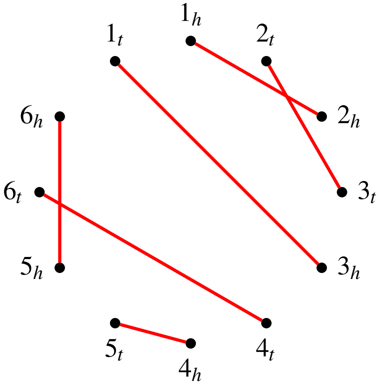

- There is an one-to-one correspondence between all perfect matchings and all genomes on the same set of genes.

Double-Cut-and-Join (DCJ) Operation

- A commonly used model for genome rearrangement
  Input: $\{p,q\}+\{s,t\}$
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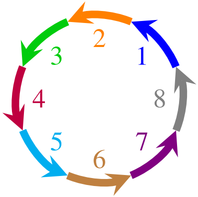

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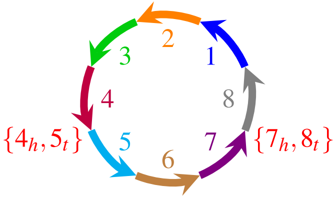

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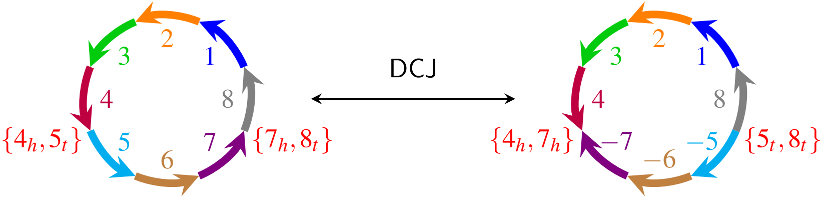

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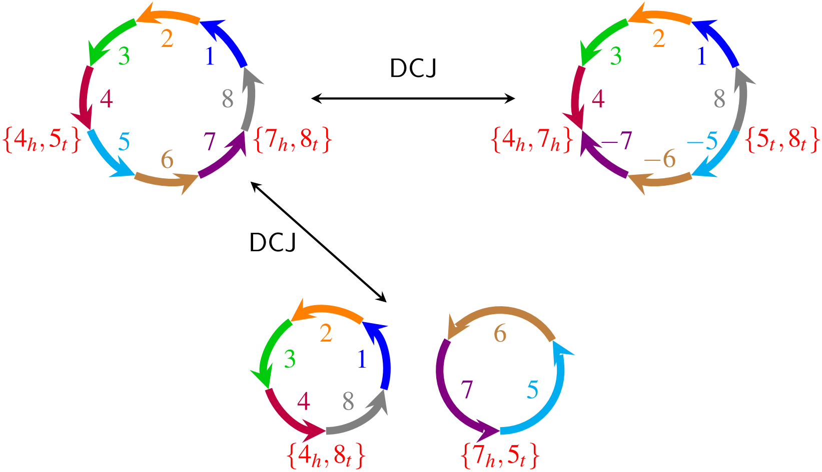

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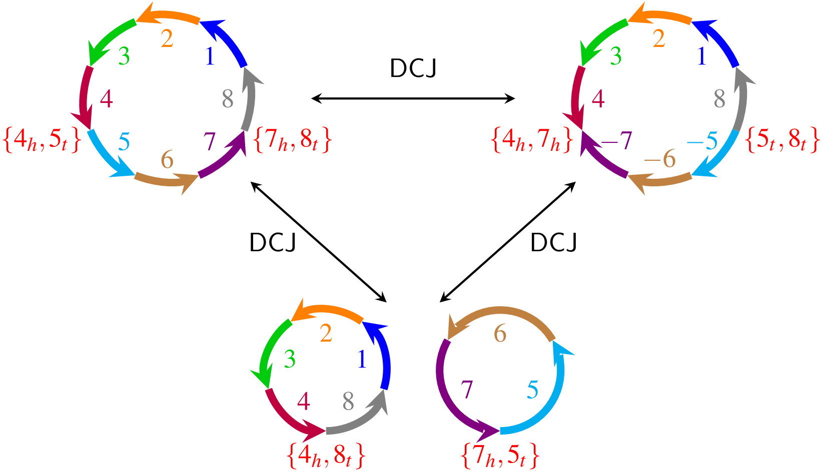

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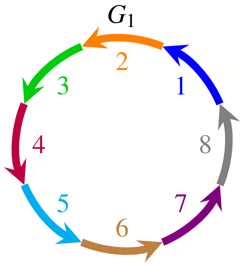

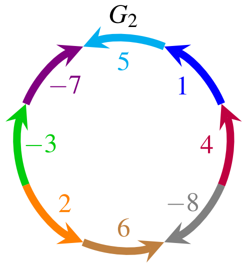

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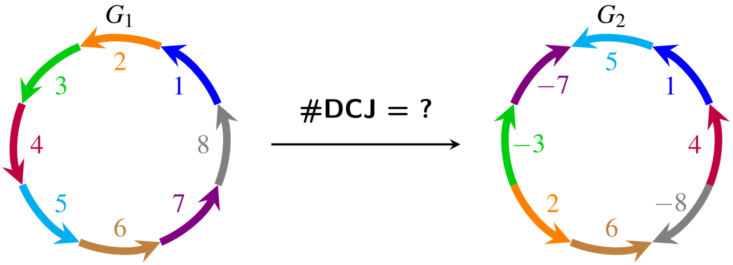

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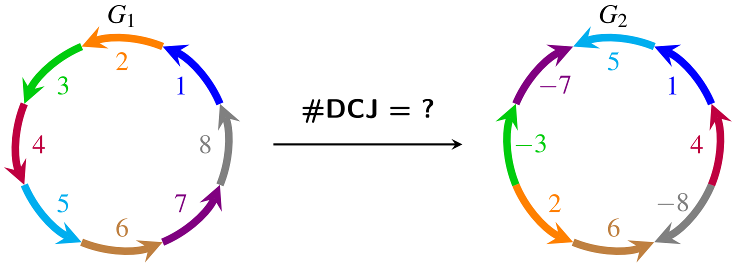
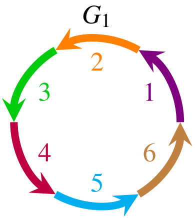
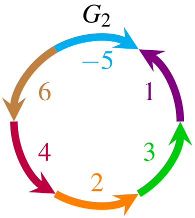


- The DCJ distance can be computed in **linear time** (*Yancopoulos et al., 2005, Bergeron et al., 2006*).

DCJ Distance Formula





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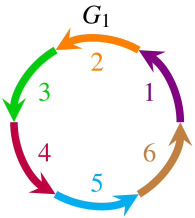

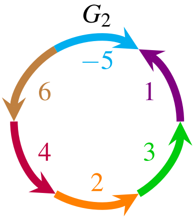

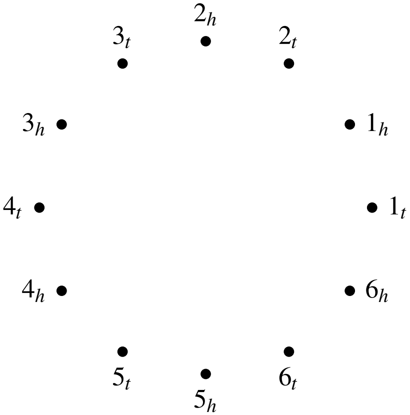

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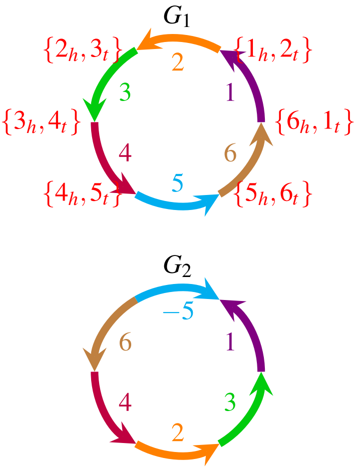

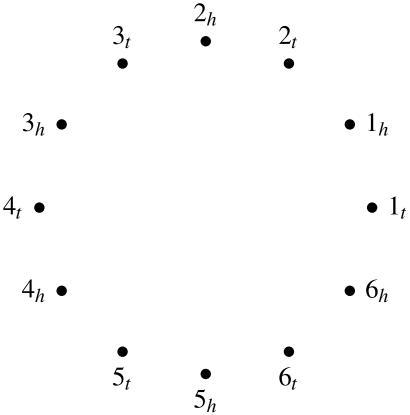

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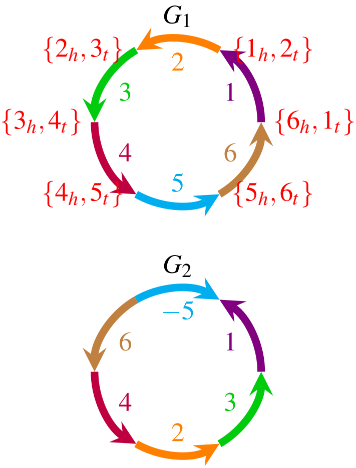

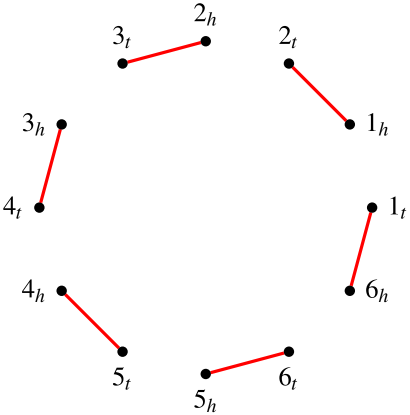

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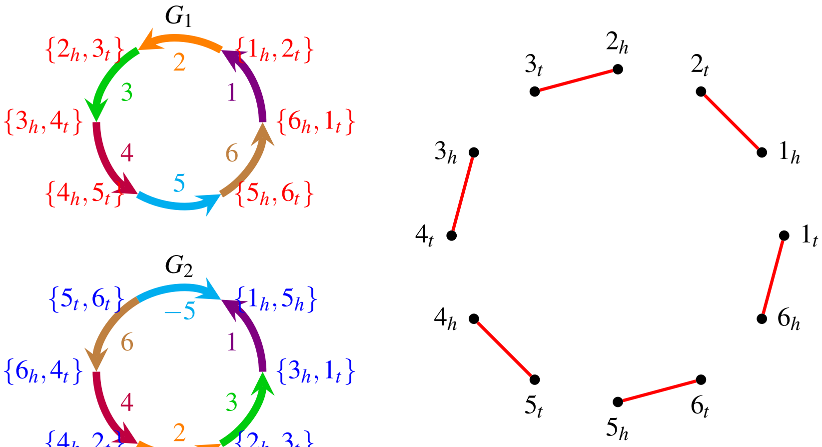

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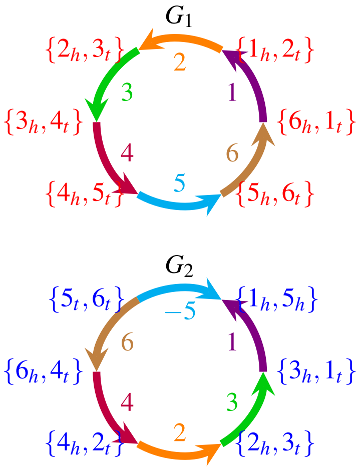

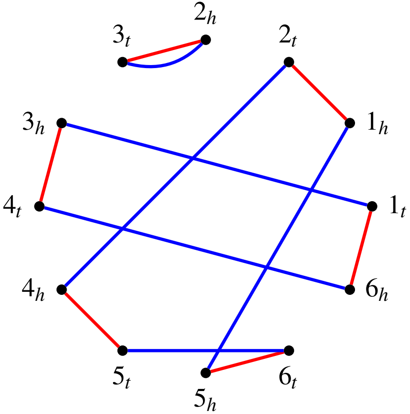

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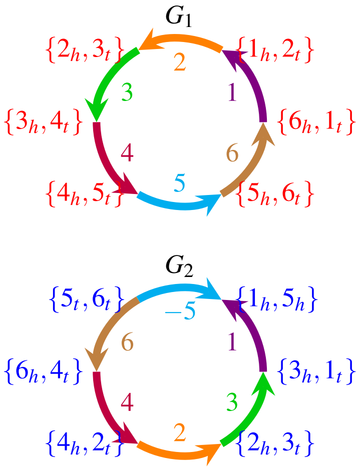

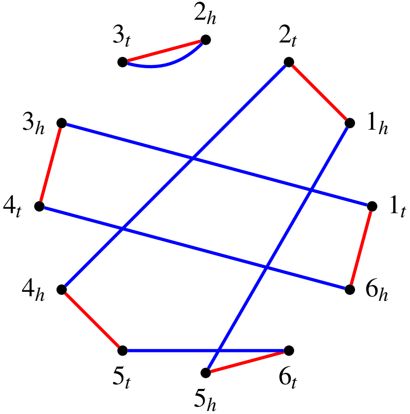

$$d(G_1, G_2) = |V|/2 - c(G_1, G_2)$$

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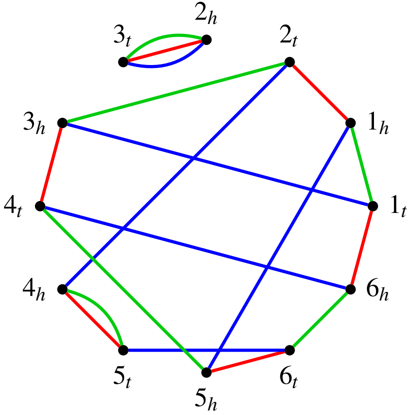

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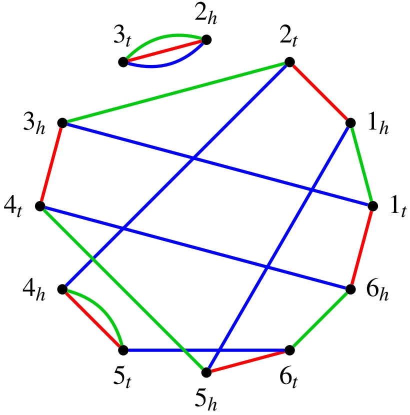

$$\max_{G} \sum_{k=1}^{3} c(G, G_k)$$

DCJ Median Problem: Current Progress

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- **Decomposition approach (***Xu et al., 2008, 2009***)**
  - **Adequate subgraph:** optimal substructure
  - **ASMedian:** fast solver using adequate subgraphs

Adequate Subgraph

- We say (G_1, G_2, G_3) is **adequate** if there exists a **certificate** matching G such that $\sum_{k=1}^{3} c(G, G_k) \geq 3 \cdot |V|/4$.

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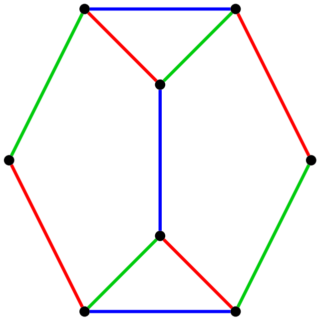

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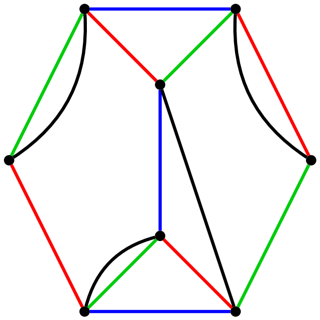

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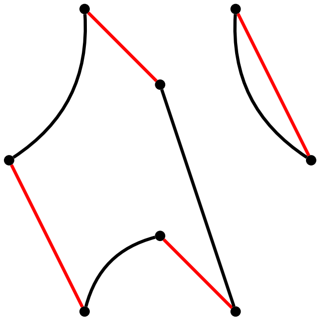

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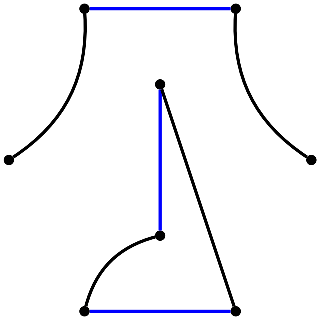

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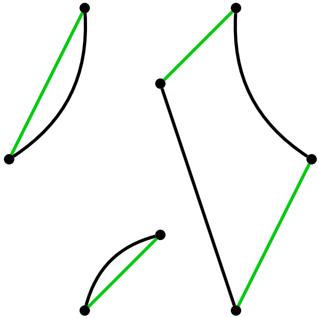

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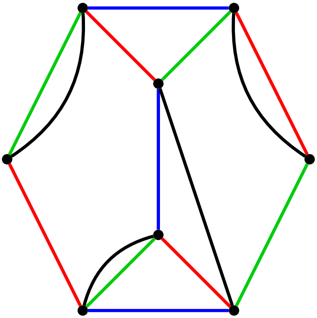

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$\sum_{k=1}^{3} c(G, G_k) = 6 \geq 3|V|/4 = 6$

Decomposition Theorem (*Xu et al., 2008*)

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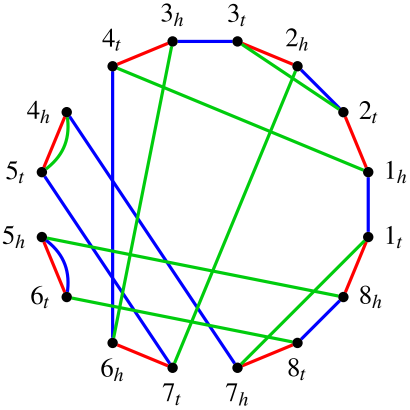

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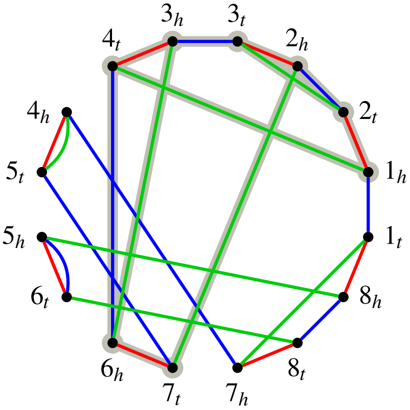

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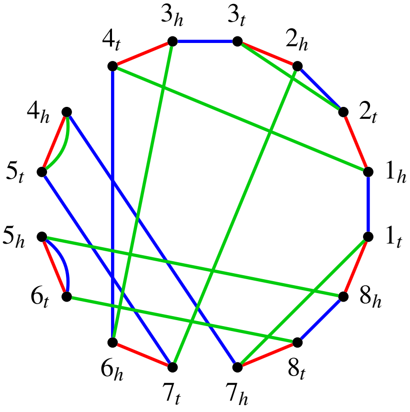

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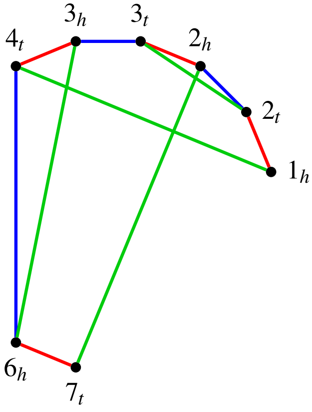

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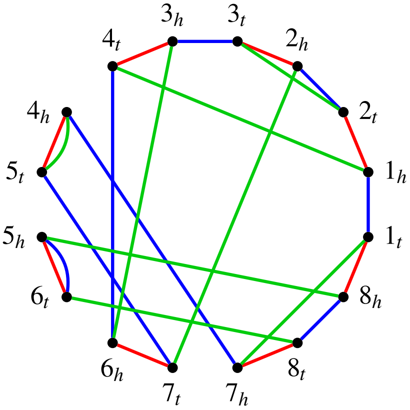

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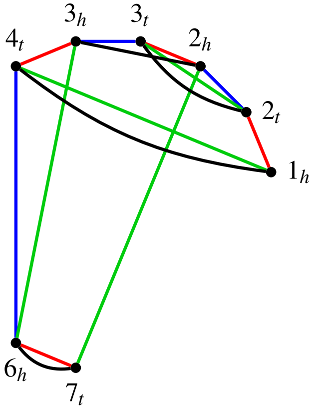

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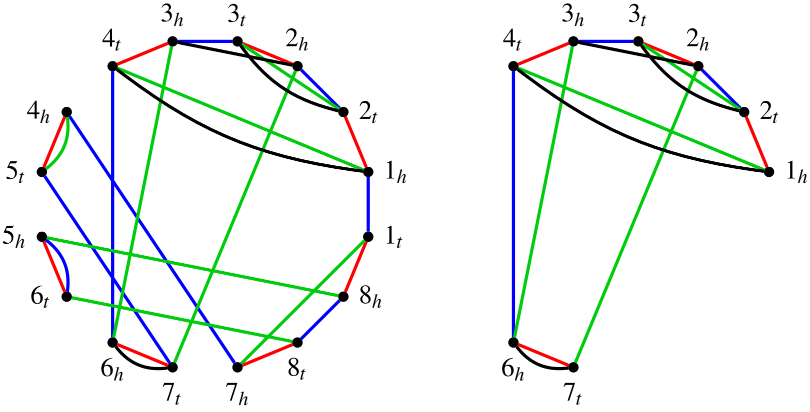

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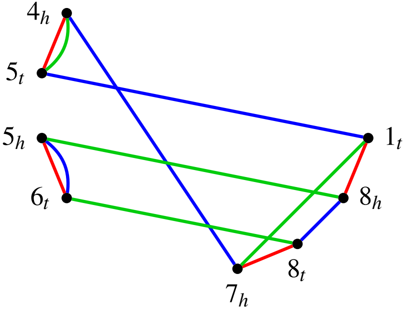

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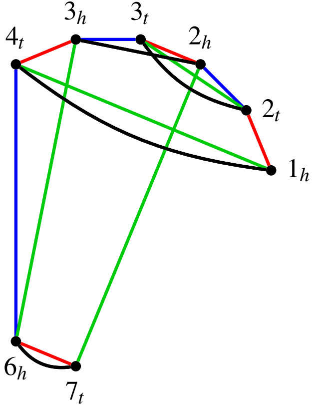

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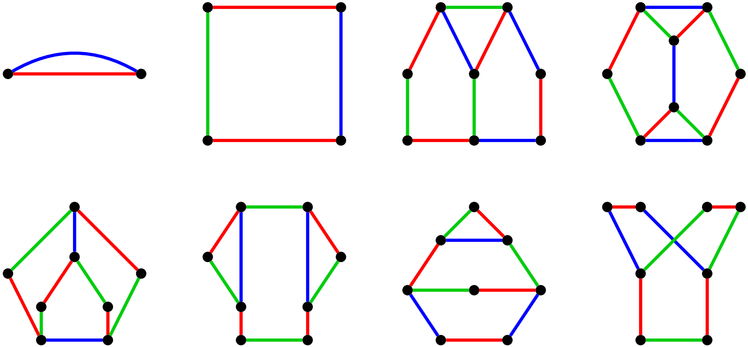

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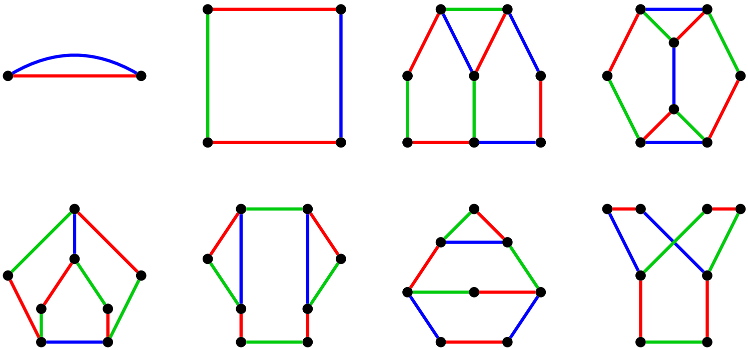

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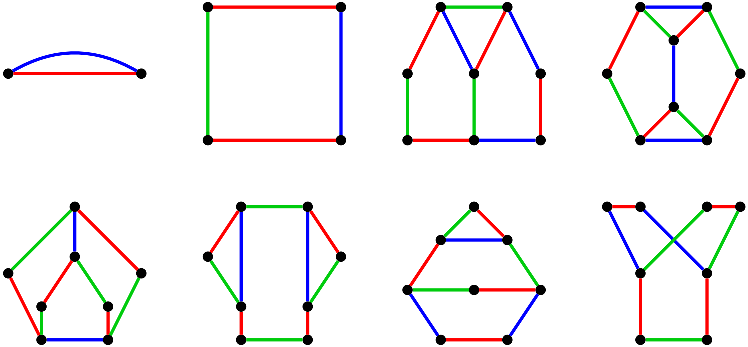

- Subgraph isomorphism problem is NP-complete.
- There are infinite many adequate subgraphs.

Our Algorithm to Find Adequate Subgraphs

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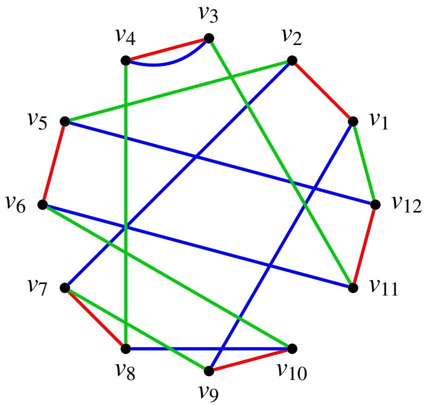

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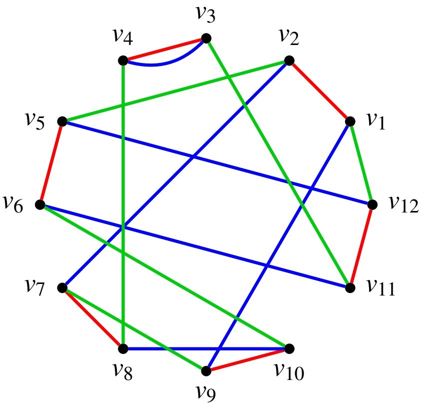

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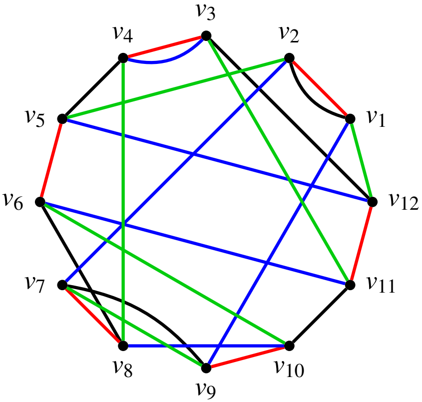

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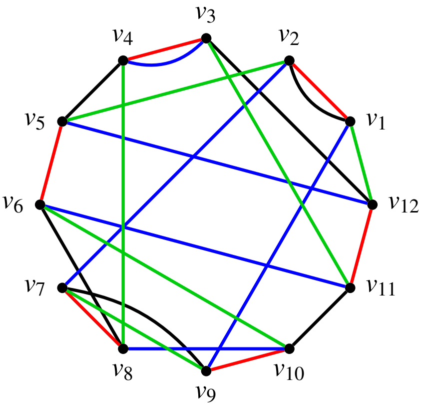

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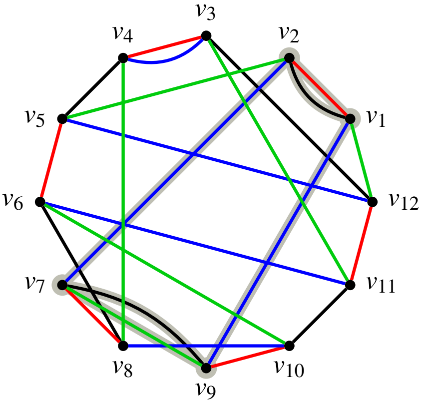

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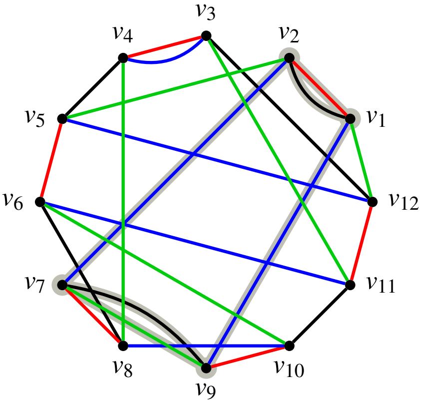

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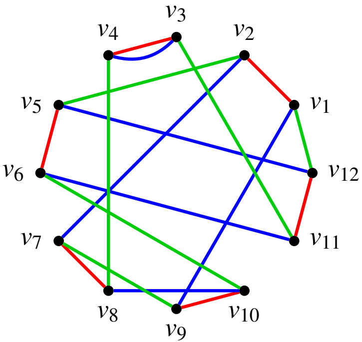
- This problem can be formulated as a **network flow** problem.

Network Construction



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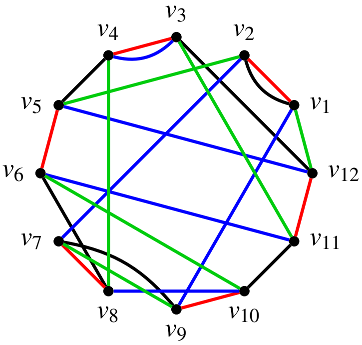

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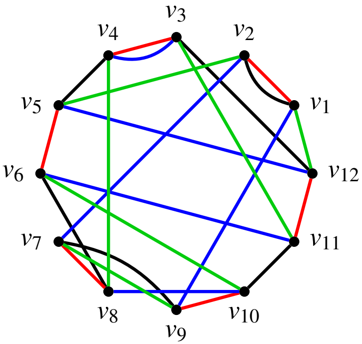

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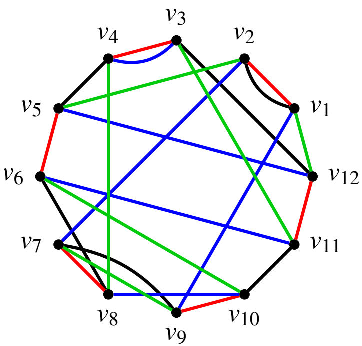

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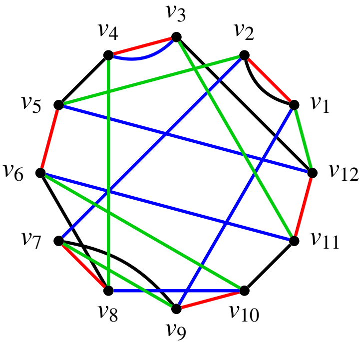

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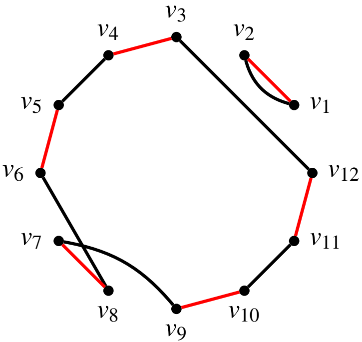

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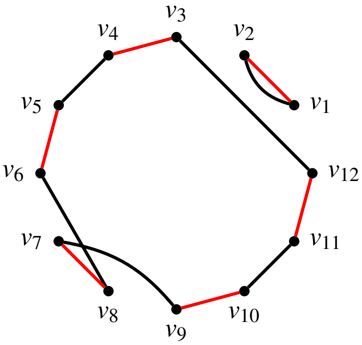

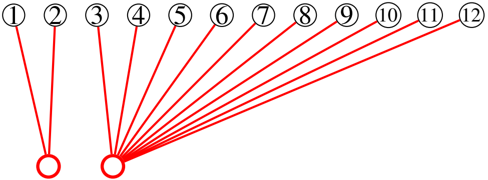

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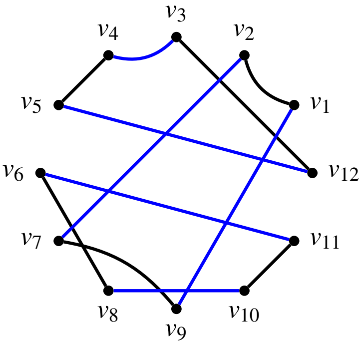

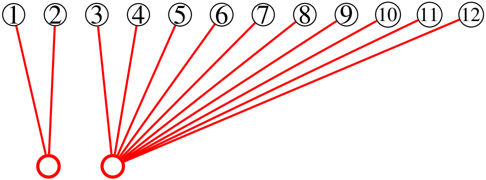

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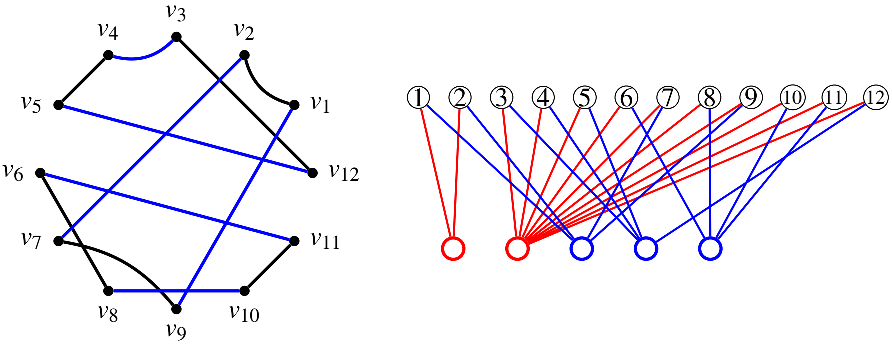

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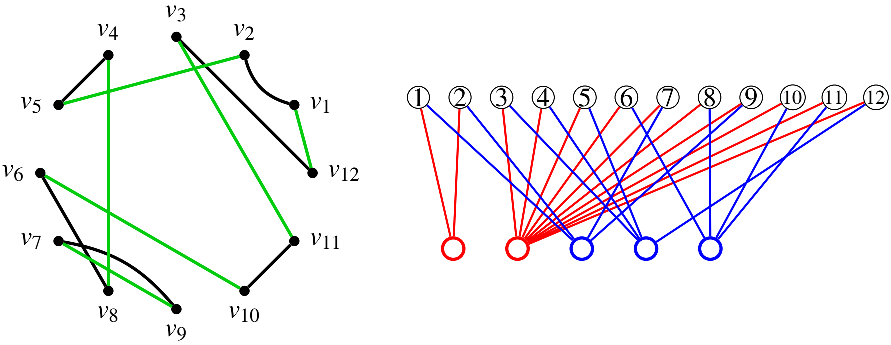

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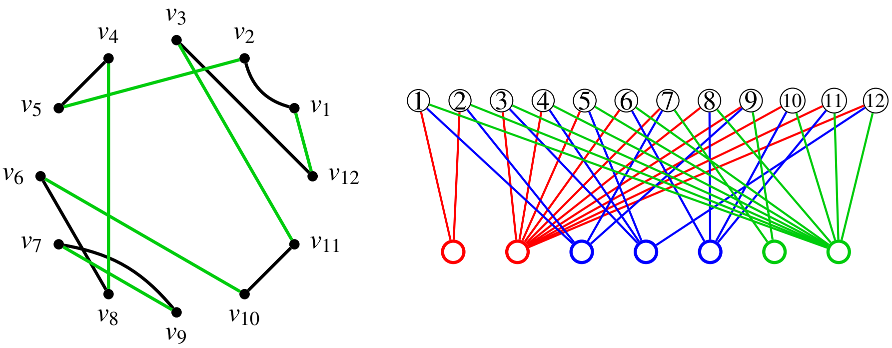

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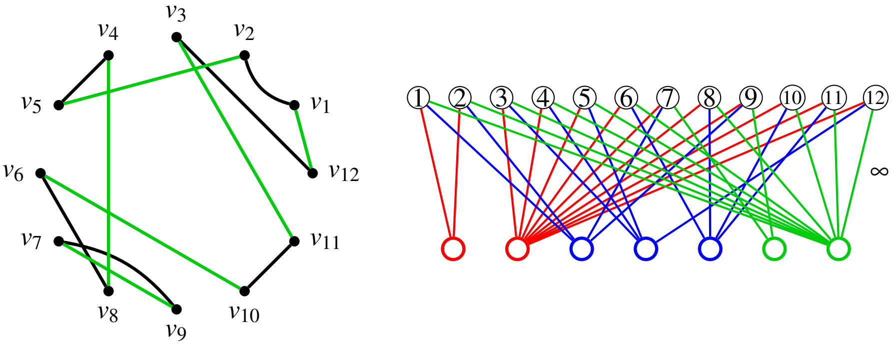

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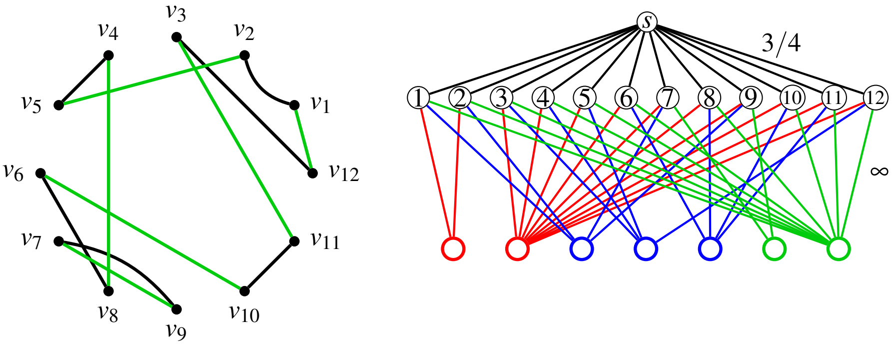

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- Add one source s, and connect it to all extremities with capacity of $3/4$.

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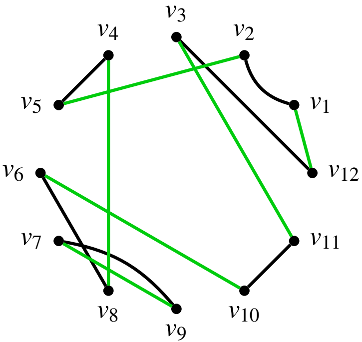

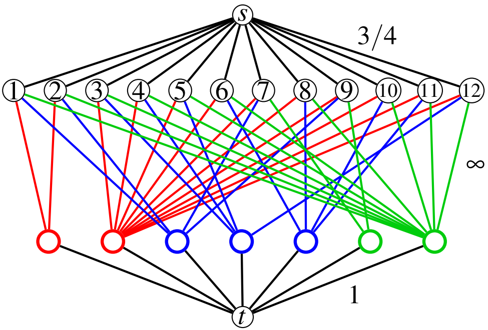

- Add one vertex for each extremity.
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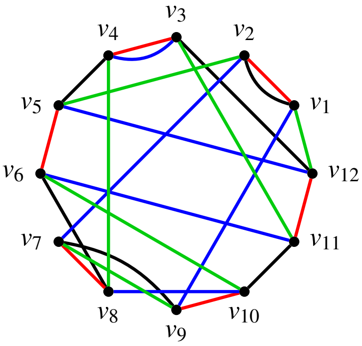
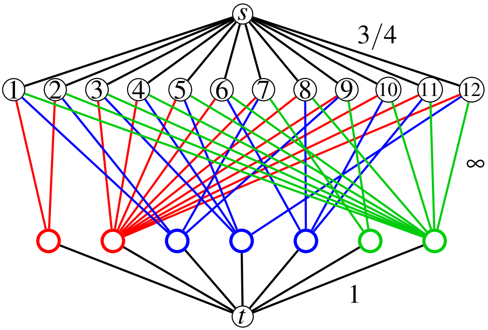
- Add one source s, and connect it to all extremities with capacity of $3/4$.
- Add one sink t, and connect it to all cycles with unit capacity.

Decide and Compute the Adequate Subgraphs





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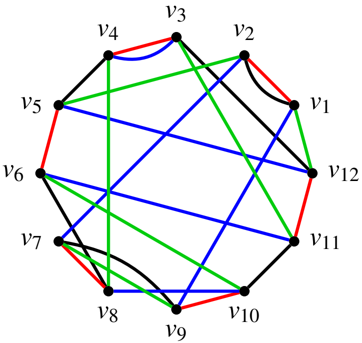

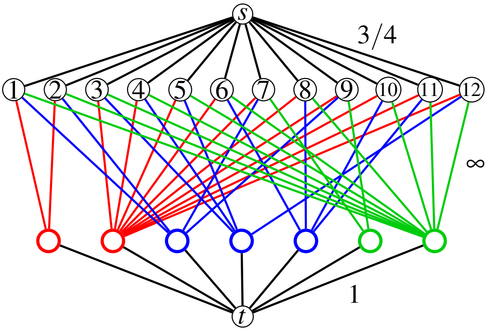

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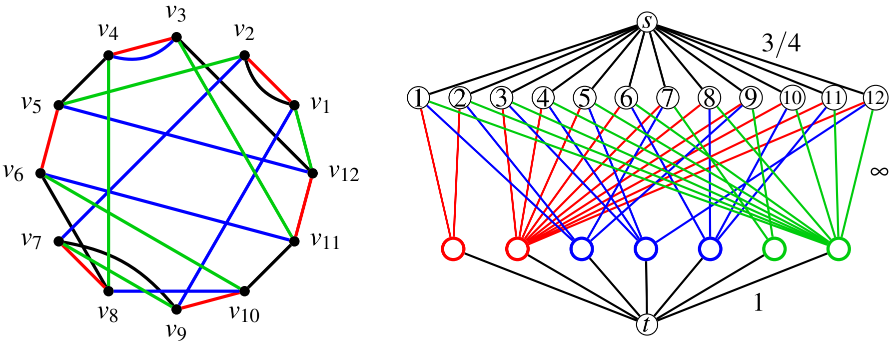

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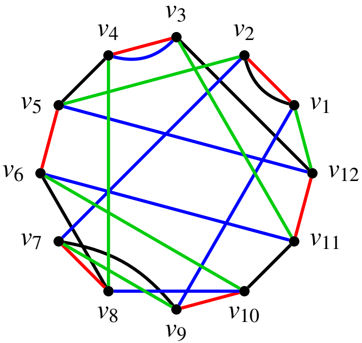

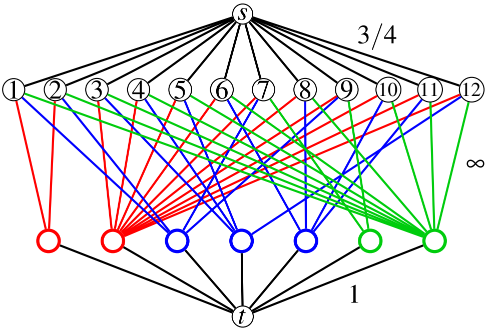

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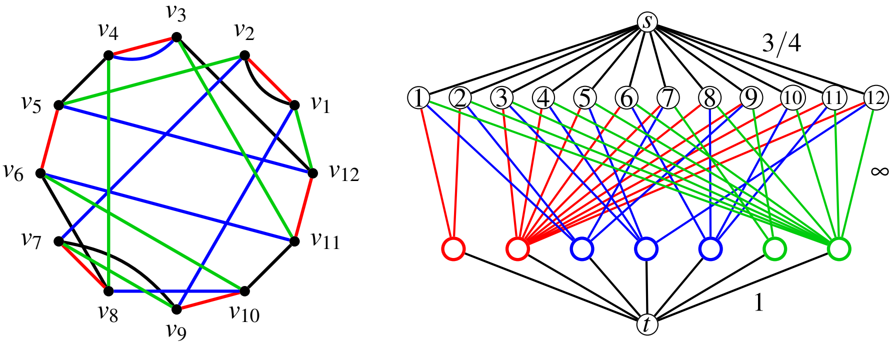

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- If such an adequate subgraph exists, we can also retrieve one from N_f.

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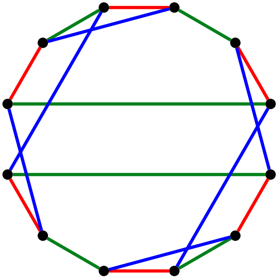

This graph is adequate, thus we can build an example of arbitray size.

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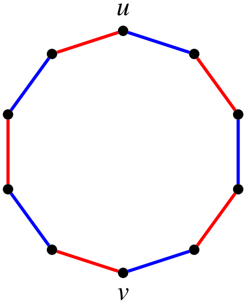

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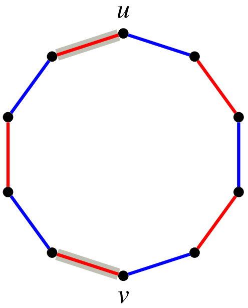

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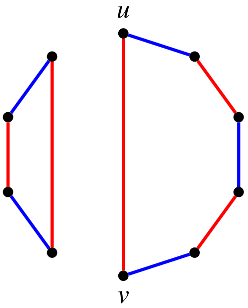

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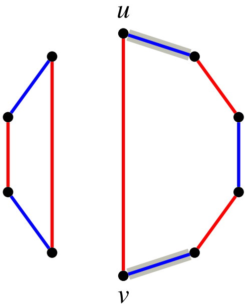

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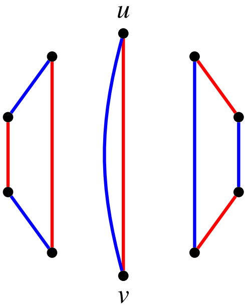

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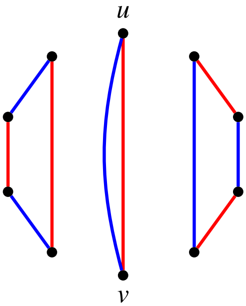

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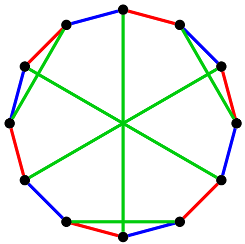

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Thank You!