Pattern Matching with Variables: A Multivariate Complexity Analysis

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Words with Coloured Holes

A word with (coloured) *holes*...

a b □ c □ c b □ b □ c a □

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$$\mathsf{a}\,\mathsf{b}\,\square\,\mathsf{c}\,\square\,\mathsf{c}\,\mathsf{b}\,\square\,\mathsf{b}\,\square\,\mathsf{c}\,\mathsf{a}\,\square$$

. . . can be repaired. . .

$$\mathsf{a}\,\mathsf{b}\,\mathsf{a} \to \square$$

$$\mathsf{c}\,\mathsf{b} \to \square$$

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A word with (coloured) *holes*...

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...can be repaired...

$$a\ b\ a \to \square$$

$$c\ b \to \square$$

...by filling in new words:

$$a\ b\ \boxed{a\ b\ a}\ c\ \boxed{c\ b}\ c\ b\ \boxed{a\ b\ a}\ b\ \boxed{c\ b}\ c\ a\ \boxed{c\ b}$$

A Special Kind of Pattern Matching

For given

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w (a word without holes),

is it possible to fill the holes of α in such a way that we obtain w?

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Example 1:

$$\alpha = \square\, \mathsf{a}\, \mathsf{a}\, \square\, \square\, \square\, \mathsf{c}\, \mathsf{b}\, \square$$

$$u = \mathsf{a}\, \mathsf{c}\, \mathsf{a}\, \mathsf{a}\, \mathsf{a}\, \mathsf{b}\, \mathsf{c}\, \mathsf{b}\, \mathsf{a}\, \mathsf{a}\, \mathsf{c}\, \mathsf{a}\, \mathsf{b}\, \mathsf{c}\, \mathsf{b}\, \mathsf{a}\, \mathsf{c}\, \mathsf{b}\, \mathsf{a}\, \mathsf{c}$$

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Example 2:

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$$w = \mathsf{c}\, \mathsf{c}\, \mathsf{b}\, \mathsf{a}\, \mathsf{a}\, \mathsf{c}\, \mathsf{c}\, \mathsf{b}\, \mathsf{c}\, \mathsf{b}\, \mathsf{c}\, \mathsf{c}\, \mathsf{b}$$

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$X \to \Sigma^+$	is a substitution	$h(x_1) := \mathtt{ab},\ h(x_2) := \mathtt{bcc}$

Pattern Matching with Variables

VPatMatch

Instance: A pattern $\alpha \in (\Sigma \cup X)^*$*, a word* $w \in \Sigma^*$*.*

Question: Does there exist a substitution h *with* $h(\alpha) = w$*?*

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Two variants:

E-VPatMatch Substitution may map variables to the empty word ε.

NE-VPatMatch Substitution can only map to non-empty words.

A Very Brief History

Three branches:

- Learning theory and Language theory (1980 - today):
  - Membership problem of Angluin's pattern languages.
  - First NE-case, later E-case.
  - Word equations, where one side is "variable-free".

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  - Baker's parameterised matching (finding repetitions in program code).
  - A. Amir, Y. Aumann, R. Cole, M. Lewenstein: function matching.
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  - R. Clifford, A. W. Harrow, A. Popa, B. Sach: generalised matching.
  - Only NE-case.
- The "real" world (?? - today):
  - Matchtest for regular expressions with backreferences.
  - Nowadays a standard tool in text editors (grep, emacs, ...) and programming language (Perl, Java, Python, ...).

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3CNF formula (without negated variables)

$$\psi = (v_1 \vee v_2 \vee v_3) \wedge (v_2 \vee v_4 \vee v_5) \wedge (v_3 \vee v_1 \vee v_3) \wedge (v_4 \vee v_1 \vee v_2)$$

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E-VPatMatch instance:

$$\alpha_\psi = x_1 x_2 x_3 \; \mathtt{b} \; x_2 x_4 x_5 \; \mathtt{b} \; x_3 x_1 x_3 \; \mathtt{b} \; x_4 x_1 x_2$$

$$w_\psi = \mathtt{a}\,\mathtt{b}\,\mathtt{a}\,\mathtt{b}\,\mathtt{a}\,\mathtt{b}\,\mathtt{a}.$$

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$$w_\psi = \mathtt{a}\,\mathtt{b}\,\mathtt{a}\,\mathtt{b}\,\mathtt{a}\,\mathtt{b}\,\mathtt{a}.$$

$\exists\, h : h(\alpha_\psi) = w_\psi$ iff ψ is "1-in-3-satisfiable".

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- we are only interested in texts of size at most 50,
- we are only interested in injective substitutions,
- in our patterns every variable occurs at most twice,
- we are only interested in patterns without any terminal symbols and we only consider substitutions of the form $h : X \to \{\mathtt{a}, \mathtt{b}, \varepsilon\}$?
  (i. e., for some u over some alphabet Γ and some $w \in \{\mathtt{a}, \mathtt{b}\}^*$, can we obtain w by replacing every $x \in \Gamma$ in u by either $\mathtt{a}$ or $\mathtt{b}$ or deleting it?)

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$\mathrm{var}(\alpha)$	is the set of variables in α	$\mathrm{var}(\alpha) = \{x_1, x_2, x_3\}$
$\lvert\alpha\rvert_x$	is the number of Occ. of x in α	$\lvert\alpha\rvert_{x_1} = 3$

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Types of VPatMatch:

- Substitutions can be erasing or must be non-erasing.

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$\lvert \Sigma \rvert$	Alphabet size.

2^3 types, 2^5 combinations of parameters $\rightarrow$ 256 versions of VPatMatch.

Research Questions

256 Questions of the following form:

Main Research Question

For any type X of VPatMatch and for any subset P of parameters, can we bound the parameters in P by constants, such that type X of VPatMatch is still NP-complete?

First Observations

Theorem (Geilke, Zilles, 2011)

If

$$|\mathrm{var}(\alpha)| \leq c \text{ or}$$
$$|w| \leq c,$$

for some constant c, then all variants of VPatMatch *are in P.*

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Observation

If

$$|\alpha|_x = 1 \text{ or}$$
$$|\Sigma| = 1,$$

then all variants of VPatMatch are in P.

The Non-injective Case

Theorem

Erasing, non-injective $\mathrm{VPatMatch}$ *is NP-complete,*

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Theorem

Erasing, non-injective $\mathrm{VPatMatch}$ *is NP-complete,*

- *even if*

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- *even if terminal-free and*

$$\begin{aligned} |h(x)| &\leq 1\,, \\ |\alpha|_x &\leq 8, \\ |\Sigma| &\leq 2\,. \end{aligned}$$

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(Non-)Erasing, non-injective VPatMatch *is NP-complete,*

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Theorem

(Non-)Erasing, non-injective VPatMatch *is NP-complete,*

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$$\begin{aligned} |h(x)| &\leq 1\,(3), \\ |\alpha|_x &\leq 2\,(2), \\ |\Sigma| &\leq 2\,(2). \end{aligned}$$

- *even if terminal-free and*

$$\begin{aligned} |h(x)| &\leq 1\,(3), \\ |\alpha|_x &\leq \not{2}\,2\,(3), \\ |\Sigma| &\leq 2\,(4). \end{aligned}$$

The Injective Case 1/2

Theorem

Let $c_1, c_2 \in \mathbb{N}$. *All injective variants of* $\mathrm{VPatMatch}$, *restricted to*

$$|h(x)| \leq c_1,$$
$$|\Sigma| \leq c_2,$$

are in P.

The Injective Case 2/2

For all other injective variants, we have NP-completeness, but the constants are a bit larger.

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Theorem

Injective, erasing or non-erasing, terminal-free or non-terminal-free VPatMatch *is NP-complete,*

- *even if*

$$\begin{aligned} |h(x)| &\leq 19, \\ |\alpha|_x &\leq 4, \end{aligned}$$

- *even if*

$$\begin{aligned} |\alpha|_x &\leq 9, \\ |\Sigma| &\leq 5. \end{aligned}$$

Further Research 1/2

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Dichotomy Result

For any variant X of VPatMatch, for any subset P of parameters and for any set C of specific bounds for the parameters in P, is the variant X of VPatMatch still NP-complete if the parameters of P are bounded by the constants in C?

Further Research 1/2

Main Research Question

For any variant X of $\mathrm{VPatMatch}$ and for any subset P of parameters, can we bound the parameters in P by constants, such that variant X of $\mathrm{VPatMatch}$ is still NP-complete?

Dichotomy Result for Erasing and Non-injective Case

Let $c_1, c_2, c_3 \in \mathbb{N}$. Erasing, non-injective $\mathrm{VPatMatch}$, restricted to

$$\begin{aligned} |h(x)| &\leq c_1, \\ |\alpha|_x &\leq c_2, \\ |\Sigma| &\leq c_3, \end{aligned}$$

is NP-Complete **if and only if** $c_1 \geq 1$, $c_2 \geq 2$, $c_3 \geq 2$.

Further Research 2/2

Parameterized Complexity

Consider the parameters ($|\,\mathrm{var}\,|$, $|\Sigma|$, ...) as parameters in terms of *parameterized complexity theory* and investigate the parameterized complexity of the corresponding parameterized problems.

Thank you very much for your attention.