Efficient Seeds Computation Revisited

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King's College London & University of Warsaw

CPM Mondello, Palermo, June 29, 2011

Why quasiperiodicity?

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every letter of the string is covered by some occurrence of the cover

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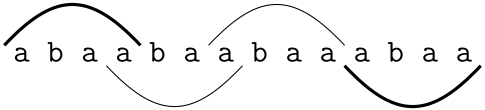

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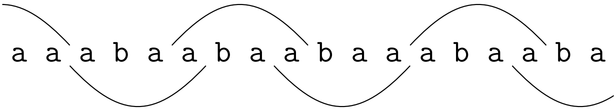

every letter of the string is covered by some occurrence of the seed, occurrences may be external

Main related problems

Problem: Cover computation
find the shortest cover (all the covers) of a string u

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Another problem: Seed computation
find the shortest seed (all the seeds) of a string

Solution:
Iliopoulos, Moore & Park (1996), $O(n \log n)$ time algorithm.

Main contributions

1. Left seeds
We introduce a natural intermediate notion between seeds and covers and give $O(n)$ time algorithms for computing the shortest left seed and the left seed array.

2. Seed array
We show how to compute the seed array in $O(n^2)$ time.

3. New (simpler) seeds computation
We present a novel approach to seed computation. Our algorithm works in $o(n \log n)$ time for some cases.

Left/right seeds

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Seed:

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is a prefix of the string, however its occurrence
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Right seed:

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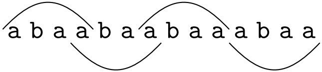

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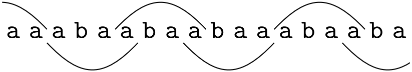
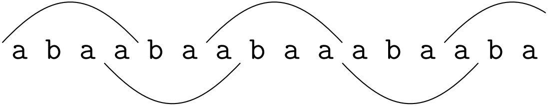


Left seed:



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Left seeds computation

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Harder problem: Left seed array
compute $\mathrm{LSeed}[1..n]$, where $\mathrm{LSeed}[i]$ is the shortest left seed of the string $u[1..i]$

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Harder problem: Left seed array
compute $\mathrm{LSeed}[1\,.\,.\,n]$, where $\mathrm{LSeed}[i]$ is the shortest left seed of the string $u[1\,.\,.\,i]$

Solution:
We present $O(n)$ time algorithms solving both the problems.

Left seeds computation

The period of a string

We say that a positive integer p is the (shortest) *period* of a string $u = u_1 \dots u_n$ (notation: $p = \mathrm{per}(u)$) if p is the smallest positive number, such that $u_i = u_{i+p}$, for $i = 1, \dots, n-p$.

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Lemma. The length of the shortest left seed of u equals:

$$\min\{\mathsf{C}[j] \ : \ \mathrm{per}(u) \le j \le |u|\}$$

where $\mathsf{C}[1 \mathinner{.\,.} n]$ is the cover array of u.

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Corollary. The left seed of a string can be computed in $O(n)$ time.

The proof

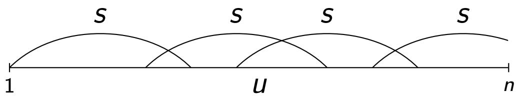
Lemma. The length of the shortest left seed of u equals:

$$\min\{C[j] \ : \ \mathrm{per}(u) \le j \le |u|\}$$

where $C[1..n]$ is the cover array of u.

Proof.

Assume that s is a left seed of u.



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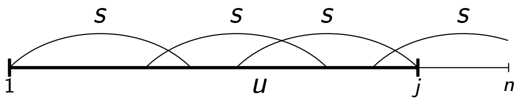

Then s is a cover of $u[1..j]$ for some j.

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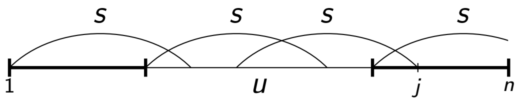

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The string u has a border $\ge n - j$, hence $\mathrm{per}(u) \le j$.

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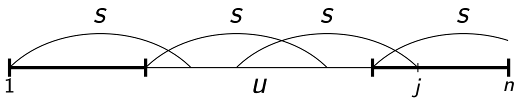

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(Recall that $\text{per}(u) + \text{border}(u) = |u|$).

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Lemma. The length of the shortest left seed of u equals:

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Proof (cont).

We have proved that the shortest left seed of u corresponds to one of the covers $\mathrm{C}[j]$ for $j \geq \mathrm{per}(u)$.

We need to show that *each value* $\mathrm{C}[j]$ for $j \geq \mathrm{per}(u)$ corresponds to some left seed of u.

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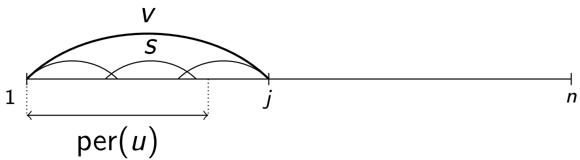
Assume that s is a cover of $v = u[1..j]$ for some $j \ge \mathrm{per}(u)$.



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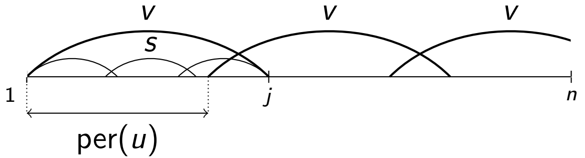

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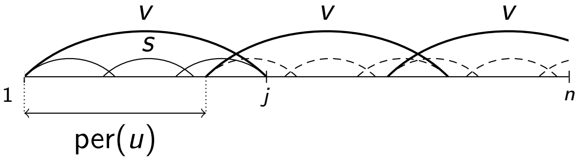

Then v is a left seed of u.

Hence, s is also a left seed of u.

Left seeds computation

Lemma. The length of the shortest left seed of u equals:

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Corollary 2. The left seed array can be computed as follows:

$$\mathsf{LSeed}[i] = \min\{\mathsf{C}[j] \ :\ \mathsf{P}[i] \le j \le i\}$$

where $\mathsf{P}[1\,.\,.\,n]$ is the period array

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The problem reduces to RMQ on the C array — $O(n)$ time algorithm.

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We present an $O(n^2)$ time algorithm.

The solution

ALGORITHM SeedArray(u)

Seed[1] := 1;
for $i := 2$ **to** n **do**
    Seed[i] := Seed[$i - 1$];
    while $u[1..i]$ does not have a seed of length Seed[i] **do**
        Seed[i] := Seed[i] + 1;
return Seed[$1..n$];

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We develop an $O(n)$ time test:

SeedsOfAGivenLength(u, k)

which checks if u has a seed of length k.

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(This test uses the suffix arrays of u.)

Seed computation

We present a new $O(n \log n)$ time algorithm for seed computation.

It can be used to check if u has the shortest seed of length $\geq m$ in $O(n \log(n/m))$ time. Hence, finding the shortest seed of length $m = \Theta(n)$ can be done in $O(n)$ time.

Links between the notions

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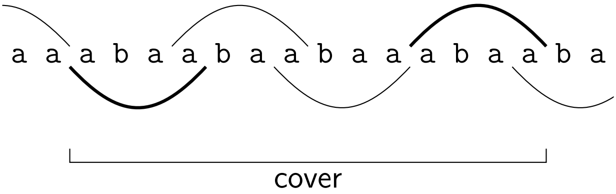

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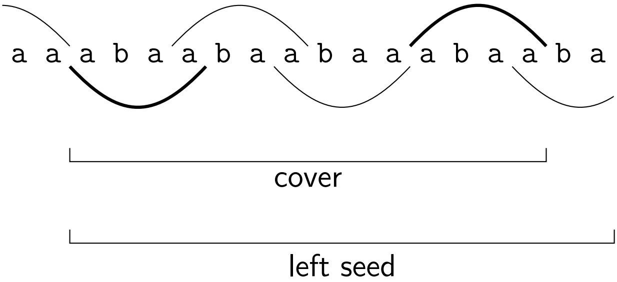

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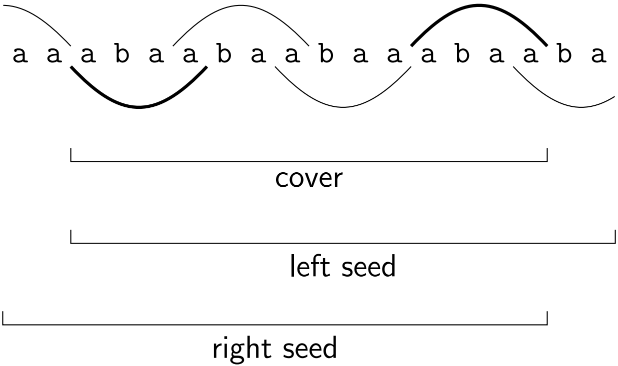

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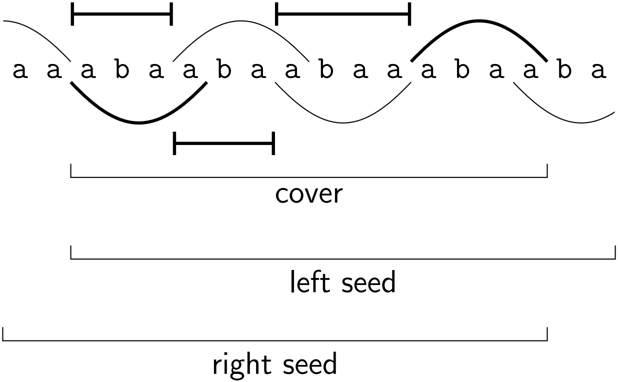
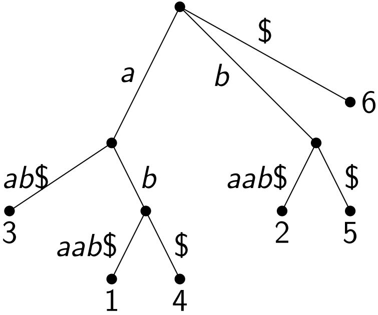


The string s is a cover of $u[i\,.\,.\,j]$ if $\mathrm{maxgap}(s) \leq |s|$.

The suffix tree



It suffices to compute maxgaps for all the explicit nodes of the suffix tree. E.g., $\text{maxgap}(ab) = \text{maxgap}(\{1, 4\}) = 3$, $\text{maxgap}(a) = \text{maxgap}(\{1, 3, 4\}) = 2$.

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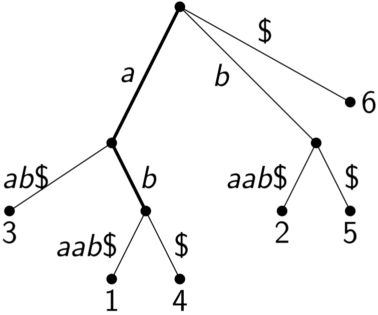

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It is easier to compute prefix maxgaps (i.e., the maxima of the maxgap values in the path from a node to the root).

Prefix maxgaps

We show how to compute all the prefix maxgaps in a path down the suffix tree in linear time.

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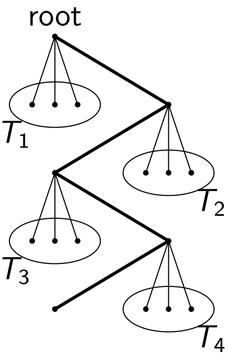

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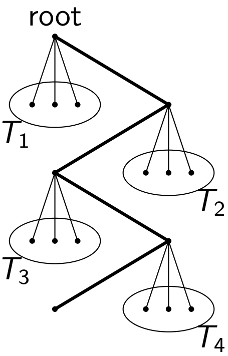

We obtain a recursive algorithm.

Each time we choose the heaviest path in the suffix tree. We obtain $O(\log n)$ levels of recursion and $O(n \log n)$ total time.

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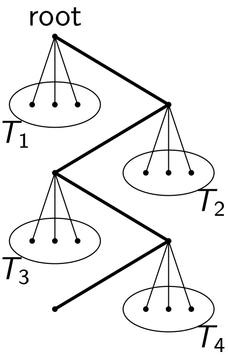

If we search for the shortest seed of length $\geq m$ then it suffices to consider several subtrees of the tree, each of size $O(n/m)$. We obtain an $O(n \log(n/m))$ time algorithm.

Summary

Cover	$O(n)$
Cover array	$O(n)$
Left seed	$\mathbf{O(n)}$
Left seed array	$\mathbf{O(n)}$
Seed	$O(n \log n)$ $[\ \mathbf{O(n \log(n/m))}\ ]$
Seed array	$\mathbf{O(n^2)}$

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Thank you for your attention!