Affine Image Matching is TC^0-complete

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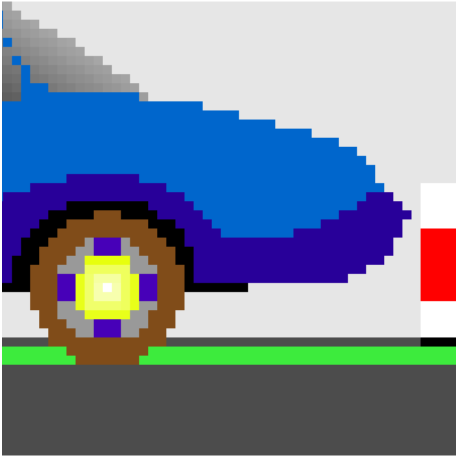
The Image Matching Problem

Remember CPM08 and CPM09 ...

The Image Matching Problem



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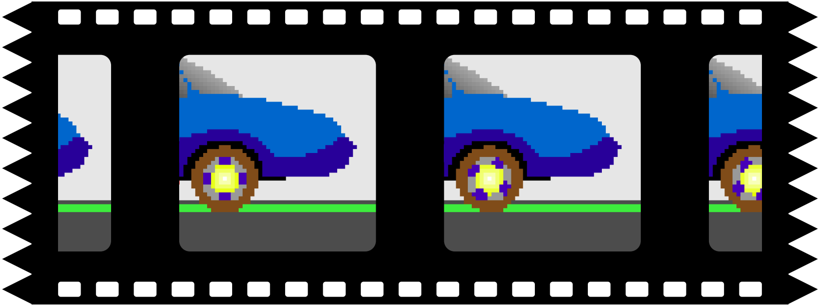

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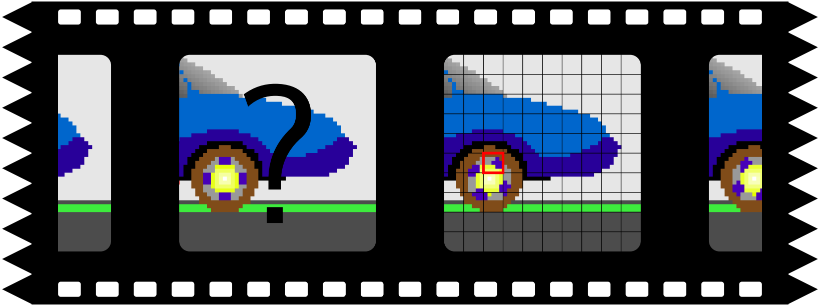


The Image Matching Problem

Given:

any subimage B

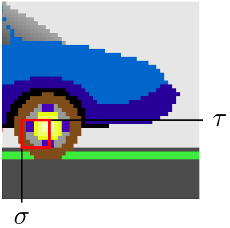
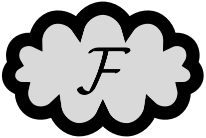


image A



The Image Matching Problem

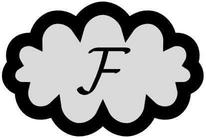
Given:

B

$\updownarrow n$

A

$\updownarrow n$



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$f(A)$

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The Image Matching Problem

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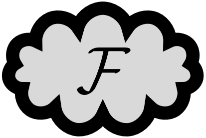
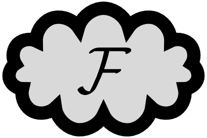


Find: $f \in \mathcal{F}$ $f(A)$  $\updownarrow n$

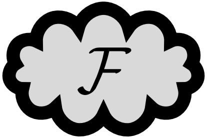
Goal: $\Delta(f(A), B) \to \min$

Transformation Class



?

Transformation Class – Affine Transformations



?

$$f(x, y) = \begin{pmatrix} a_1 & a_2 \\ a_4 & a_5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_3 \\ a_6 \end{pmatrix}$$

Discrete Problem

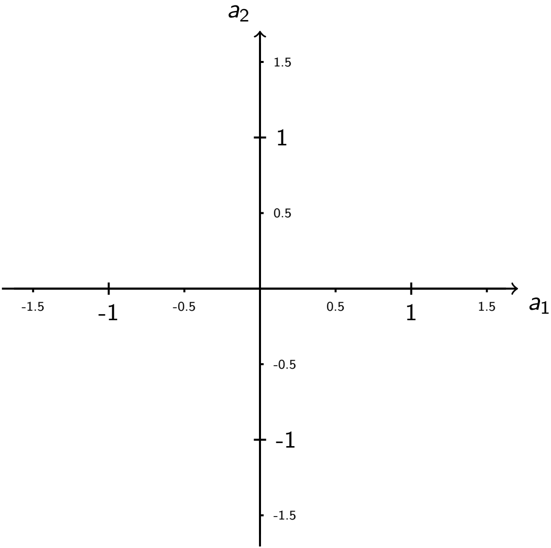
Consider $D(A) = \{f(A) \mid f \in \mathcal{F}\}$

Discrete Problem

For  $D(A)$ contains

These are only 1250 out of hundreds and thousends!

Our Characterization [MFCS08]



Consider all points

$p = (a_1, a_2, a_3, a_4, a_5, a_6)^T$

of the $\mathbb{R}^6$.

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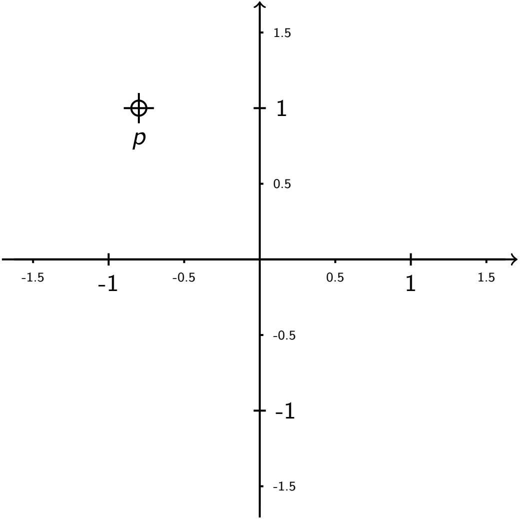

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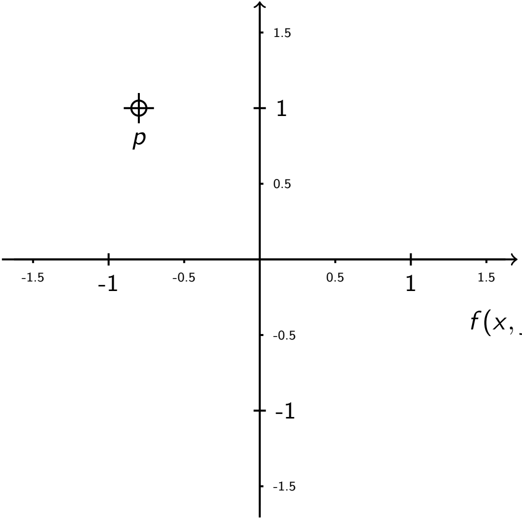

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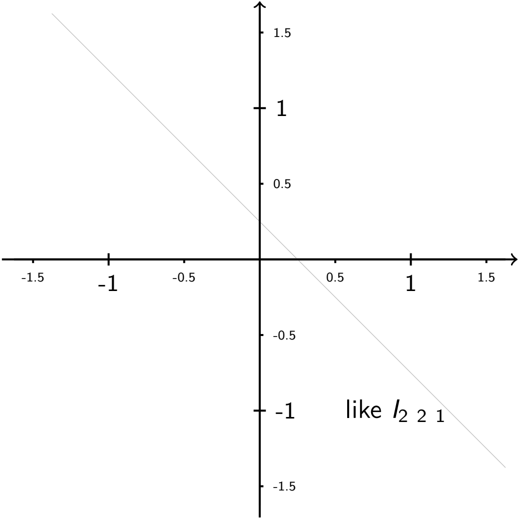

There are hyperplanes

$$I_{ijk} : ia_1 + ja_2 + a_3 = k - 0.5$$
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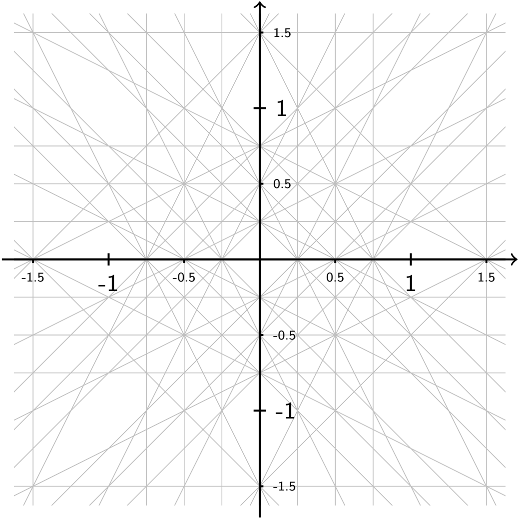

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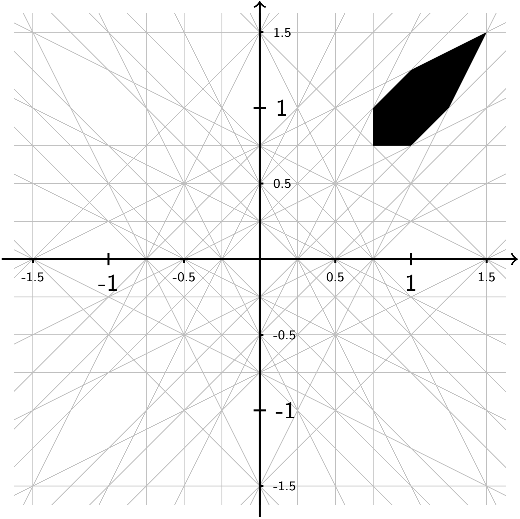

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cutting $\mathbb{R}^6$ into a set $\mathcal{A}_n$ of convex regions.

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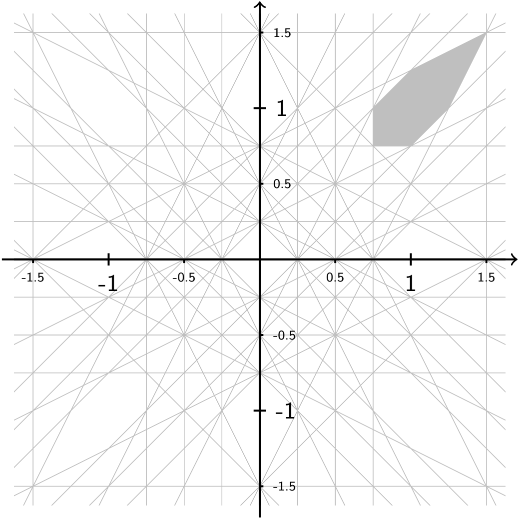

Theorem

Let $\varphi \in \mathcal{A}_n$ and let

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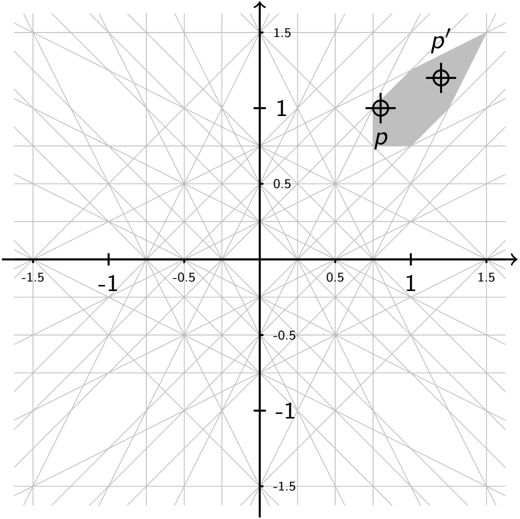

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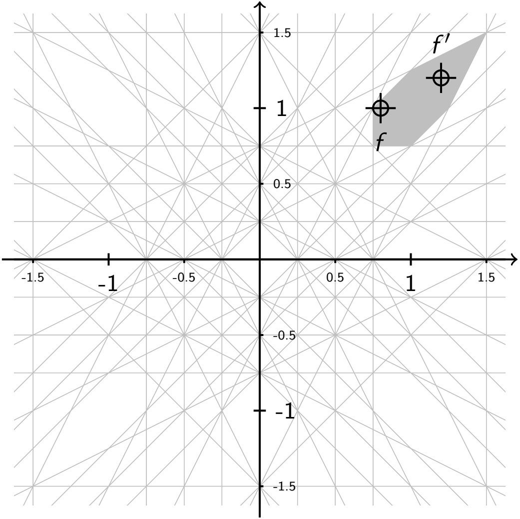

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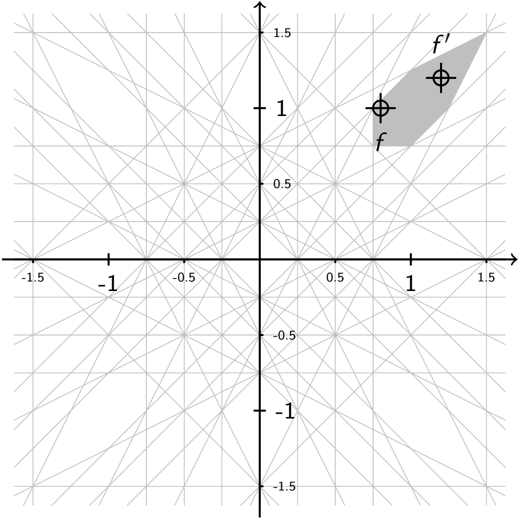

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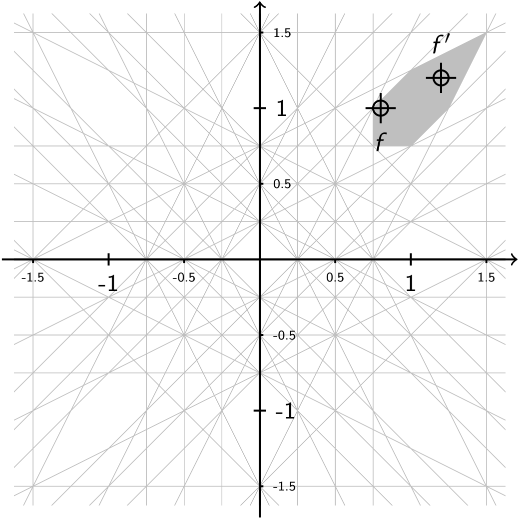

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Theorem

$|\mathcal{A}_n| \in O(n^{18})$.

A sequential Affine Image Matching Algorithm [MFCS08]

Input: Images A and B of size n

Output: $f \in \mathcal{F}$ with minimum distortion $\Delta = (f(A), B)$

1.
2.
3.
4.
5.
6.

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	overall time	$O(n^{18})$ time

It was open for a long time, whether Affine Image Matching is polynomial time solveable.

Recently raised Questions

1. What is the "exact" complexity class of Affine Image Matching?
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New Result

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Affine Image Matching is TC^0*-complete.*

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$$\mathrm{AC}^0 \subset \mathrm{TC}^0 \subseteq \mathrm{NC}^1 \subseteq \ldots \subseteq \mathsf{L} \subseteq \mathsf{P}$$

The Complexity Class TC^0

All functions $f : \{0,1\}^* \to \{0,1\}^*$ that can be computed by

1. a family of circuits $C_1, C_2, \ldots, C_n, \ldots$ that contain
2. and-gates, and with unbounded fan-in and
3. where every circuit has constant depth and
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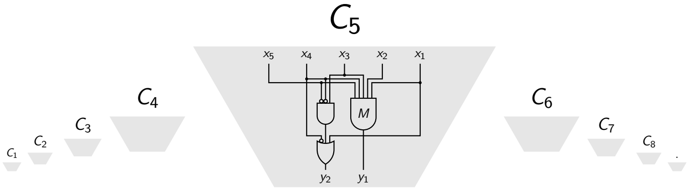

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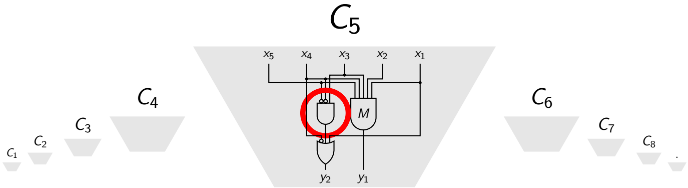

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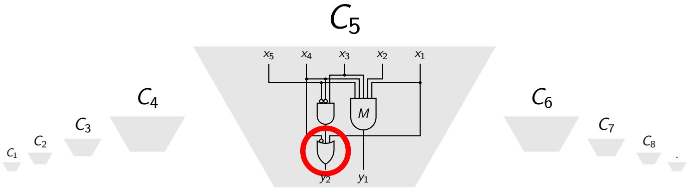

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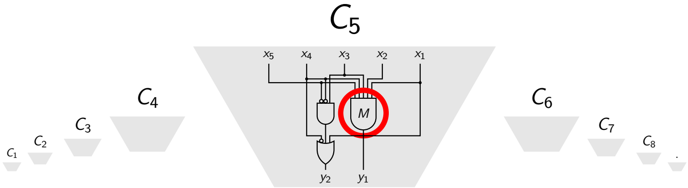

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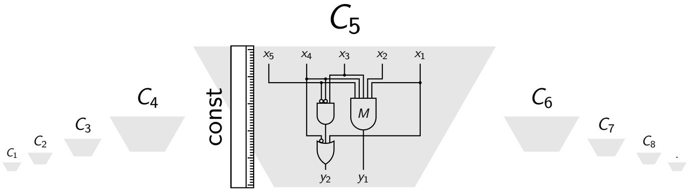

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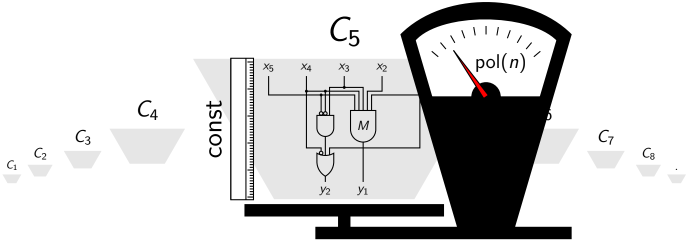

The Complexity Class TC^0

majority function (TC^0-complete)

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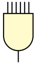
integer addition, multiplication (TC^0-complete), division (TC^0-complete), ...



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repeated integer addition, i.e., $\sum$-operations (TC^0-complete)

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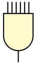

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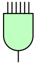
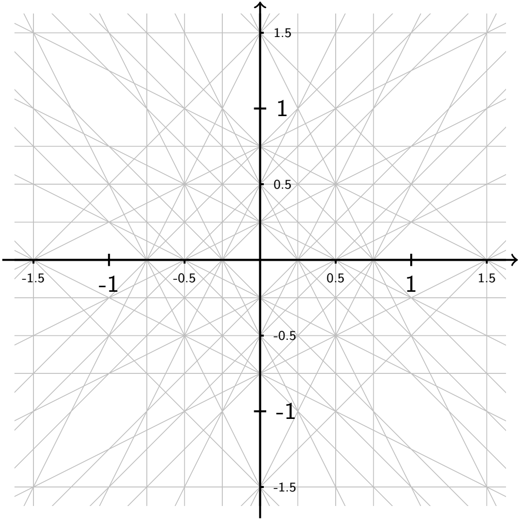
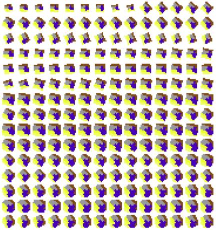


However it is unknown how $\mathcal{A}_n$ can be computed in TC^0.

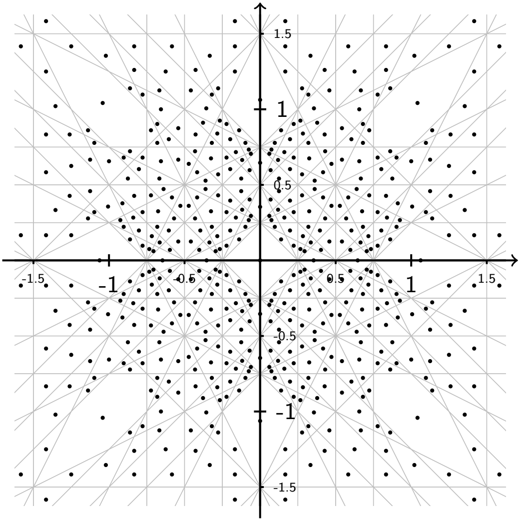


To get $D(A)$



$\vdots$

A TC^0 Affine Image Matching Approach

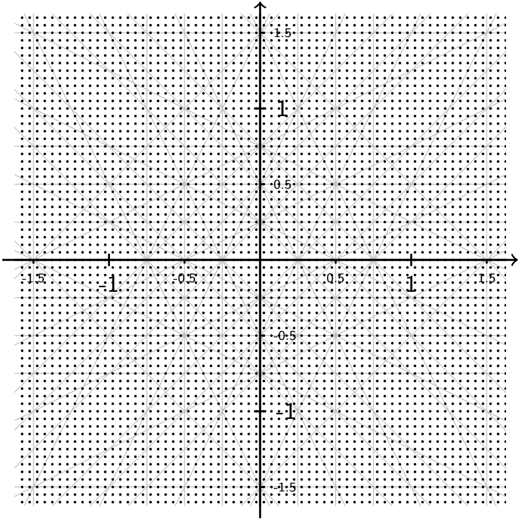


To get $D(A)$

⋮

the old algorithm processes all these points.

A TC^0 Affine Image Matching Approach



However, consider:

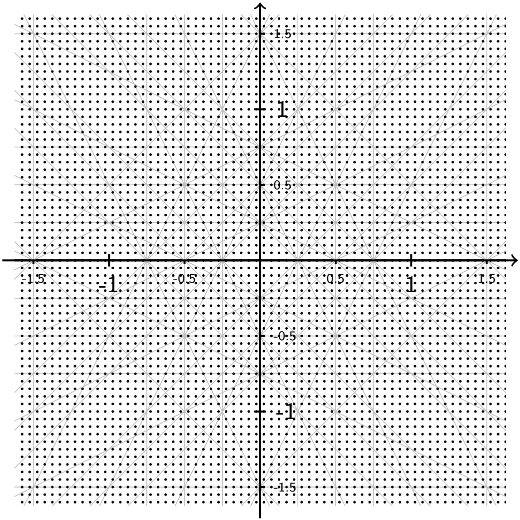
Definition

$\mathcal{G}_n$, the grid of points

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = 10^{-7} n^{-7} \cdot \begin{pmatrix} t_1 + 0.5 \\ t_2 \\ t_3 \\ t_4 \\ t_5 + 0.5 \\ t_6 \end{pmatrix}$$

for all $t_1, \ldots, t_6$ in $\{-10^{12} n^{13}, \ldots, 10^{12} n^{13}\}$.

A TC^0 Affine Image Matching Approach



Theorem

1. $|\mathcal{G}_n| \in O(n^{78})$.
2. *Every point $p = (a_1, \ldots, a_6) \in \mathcal{G}_n$ fulfills $a_1 a_5 \neq a_2 a_4$.*
3. *For every $\varphi \in \mathcal{A}_n$ there is a grid point p with $p \in \varphi$.*

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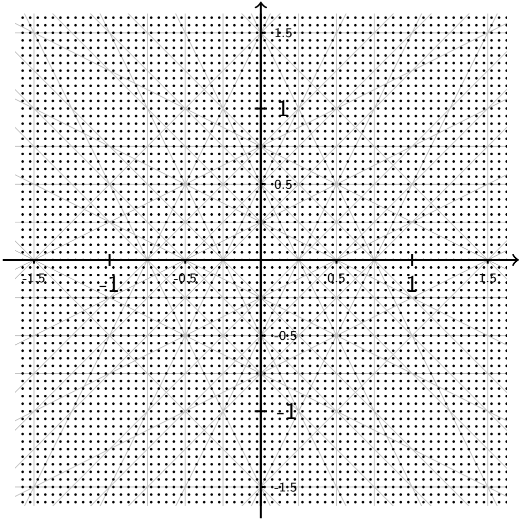

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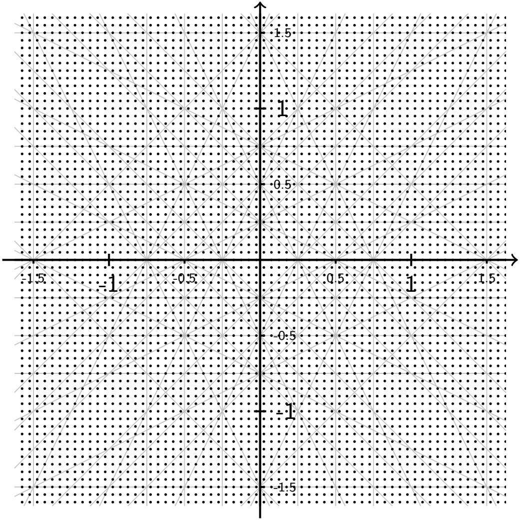
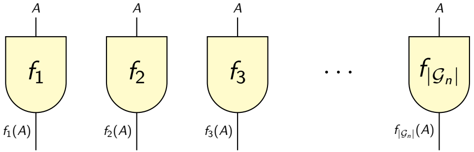
A TC^0 Affine Image Matching Approach



Theorem

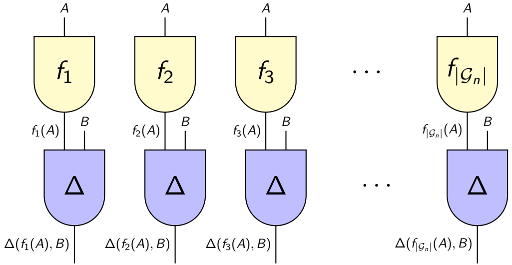
1. $|\mathcal{G}_n| \in O(n^{78})$.
2. *Every point* $p = (a_1, \ldots, a_6) \in \mathcal{G}_n$ *fulfills* $a_1 a_5 \neq a_2 a_4$.
3. *For every* $\varphi \in \mathcal{A}_n$ *there is a grid point* p *with* $p \in \varphi$.

A TC^0 Affine Image Matching Approach



essentially arithmetics, constant depth, $O(n^{78})$ times polynomially many processors

A TC^0 Affine Image Matching Approach



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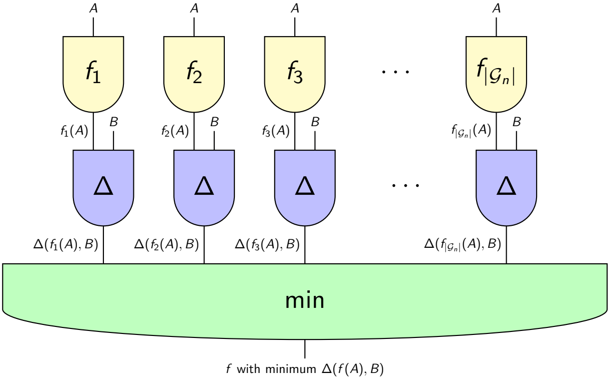

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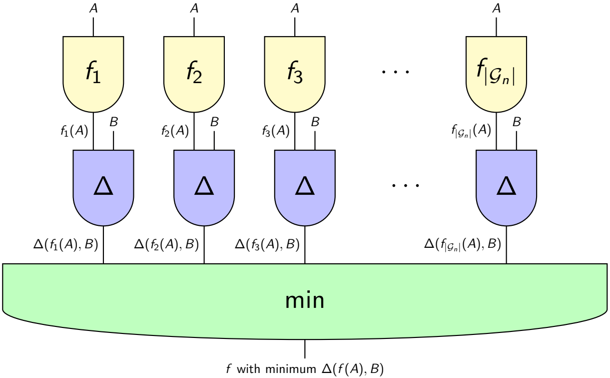

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minimum computation, constant depth, polynomially many processors

Hence, Affine Image Matching is in TC^0.

The TC^0-completeness of Affine Image Matching

Consider the TC^0-complete problem

$$\texttt{Majority} = \left\{ s \in \{0,1\}^* \;\middle|\; \sum_{i=1}^{|s|} s_i \geq \left\lfloor \frac{|s|}{2} \right\rfloor \right\}$$

The TC^0-completeness of Affine Image Matching

string s

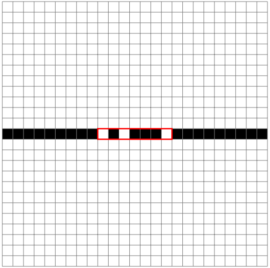
0	1	0	1	1	1	0

The TC^0-completeness of Affine Image Matching

string s

0	1	0	1	1	1	0

image A_s



The TC^0-completeness of Affine Image Matching

string s

0	1	0	1	1	1	0

image A_s

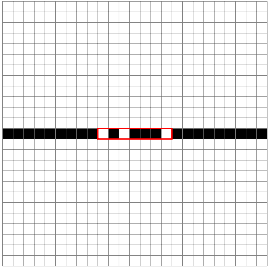
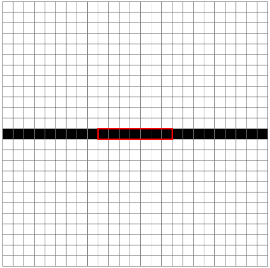


image B_s



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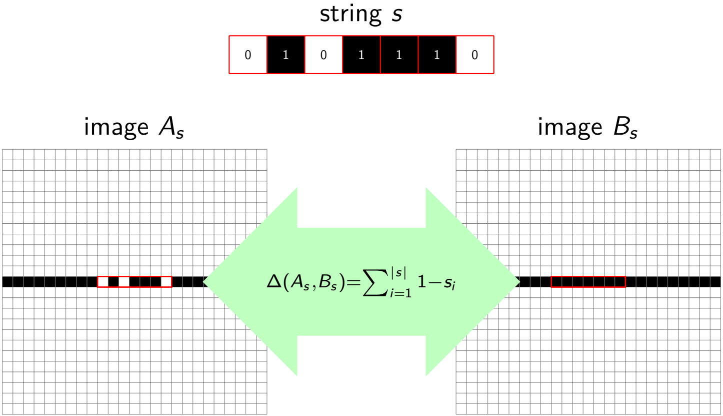
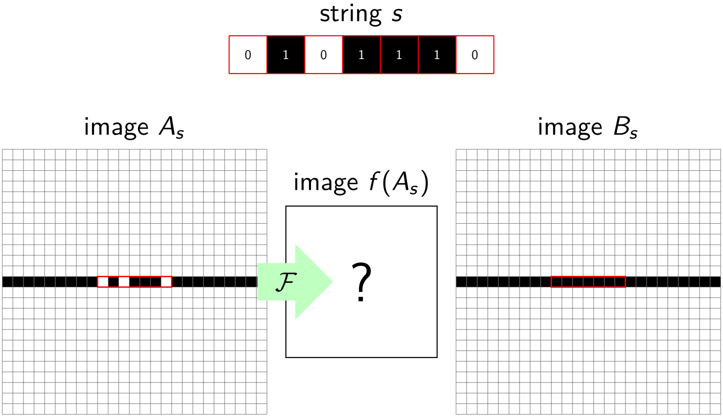


String s is in `Majority` iff $\Delta(A_s, B_s) \leq \left\lfloor \frac{|s|}{2} \right\rfloor$.

The TC^0-completeness of Affine Image Matching



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0	1	0	1	1	1	0

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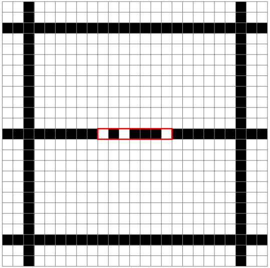
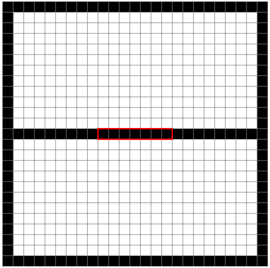
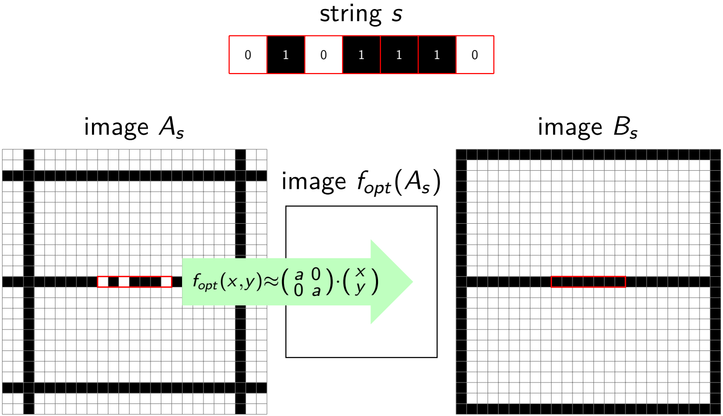


image B_s



The TC^0-completeness of Affine Image Matching



$$a = \frac{16|s|}{16|s|-7}$$

The TC^0-completeness of Affine Image Matching

string s

0	1	0	1	1	1	0

image $f_{opt}(A_s)$

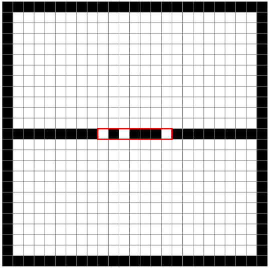
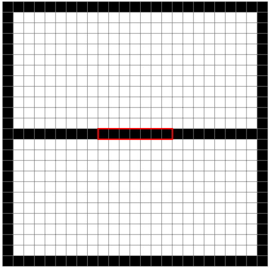


image B_s



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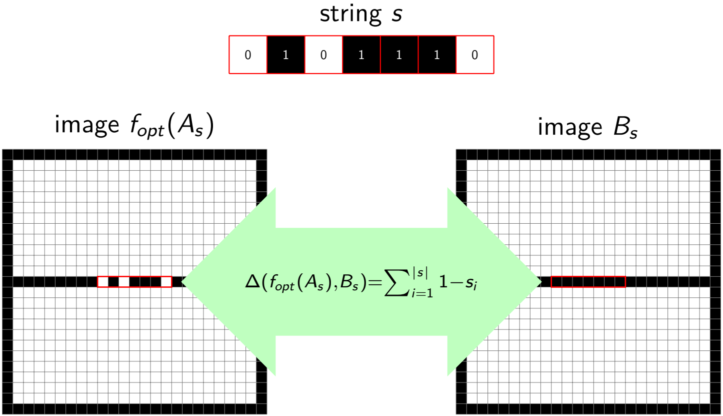
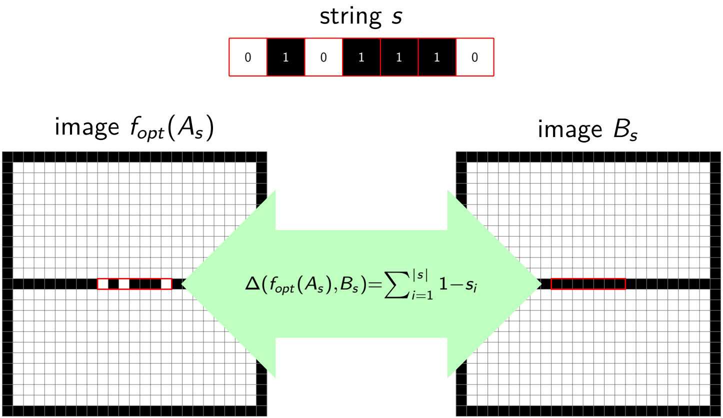


String s is in `Majority` iff $\Delta(f_{opt}(A_s), B_s) \leq \left\lfloor \frac{|s|}{2} \right\rfloor$.

The TC^0-completeness of Affine Image Matching



Because A_s and B_s can be computed easily,
Affine Image Matching is hard in TC^0.

Conclusions

- New insights into the structur of affine image transformations.
- Projective Image Matching is TC^0-complete.
- TC^0-approach does not apply to weaker transformation classes in a straight-forward manner.
- TC^0-hardness remains even for the weakest cases like Scaling or Rotation Image Matching.

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End of Talk

Thank you for your attention!