New Complexity Bounds for Image Matching under Rotation and Scaling

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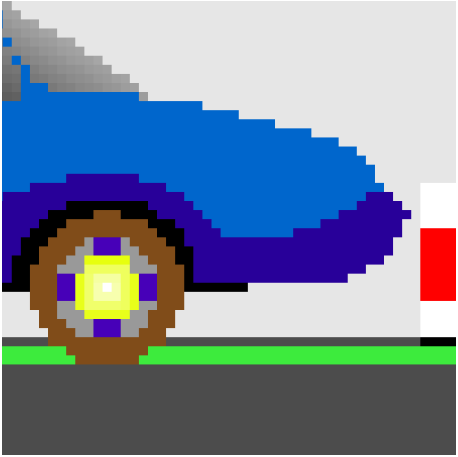
The Image Matching Problem

Remember last year's CPM ...

The Image Matching Problem



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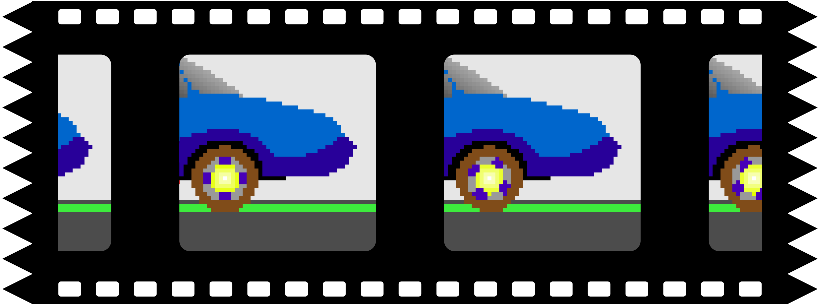

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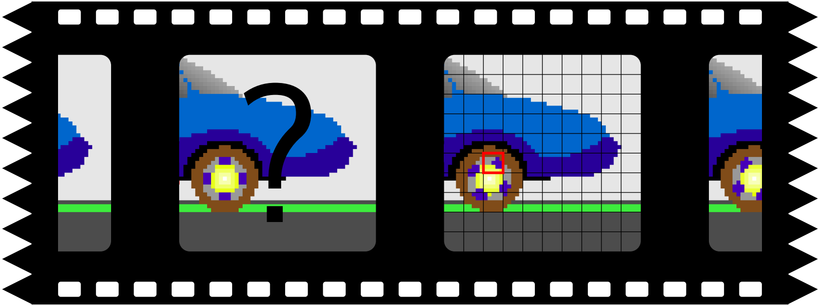


The Image Matching Problem

Given:

any subimage B

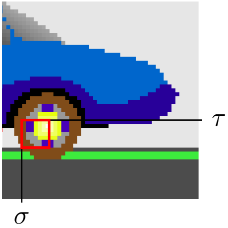
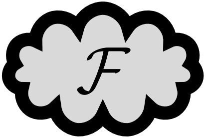
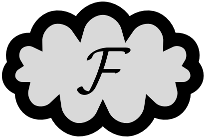


image A



The Image Matching Problem

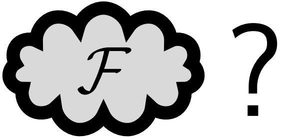
Given: B  $\updownarrow n$ A  $\updownarrow m$



Find: $f \in \mathcal{F}$ $f(A)$ 

Goal: $\Delta(f(A), B) \rightarrow \min$

Transformation Classes



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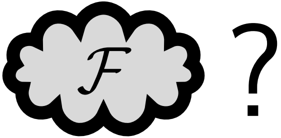


Rotation - $\mathcal{F}_{\mathtt{r}}$

$$f(x,y) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Scaling and Rotation - $\mathcal{F}_{\mathtt{sr}}$

$$f(x,y) = \begin{pmatrix} s\cos\phi & s\sin\phi \\ -s\sin\phi & s\cos\phi \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Continuous Problem?

Image Matching seems to be a continuous problem by $\mathcal{F}_{\mathtt{sr}}$.

Continuous Problem?

But it is not!

Continuous Problem?

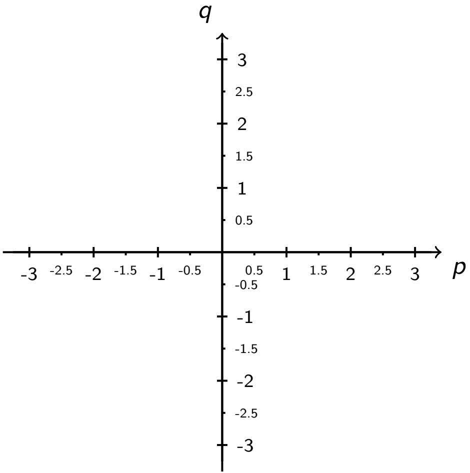
$D(A) = \{f(A) \mid f \in \mathcal{F}_{\mathtt{sr}}\}$ is always finite.

Continuous Problem?

For  $D(A)$ contains

... only 1250 out of hundreds and thousands!

Our Characterization



Let $p = s \cos \phi$ and $q = s \sin \phi$,

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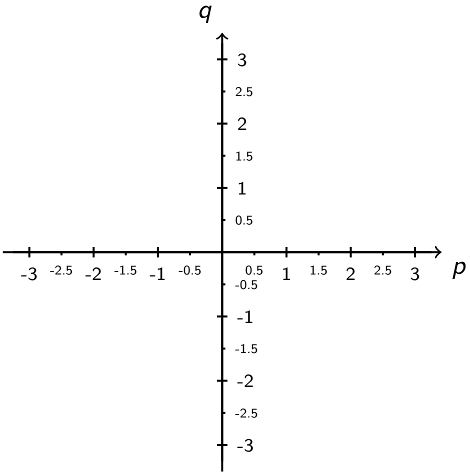

Let $p = s\cos\phi$ and $q = s\sin\phi$, then (p, q) stands for $f(p, q) \in \mathcal{F}_{\mathtt{sr}}$:

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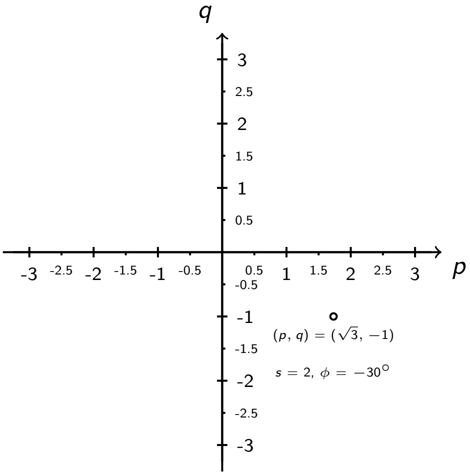

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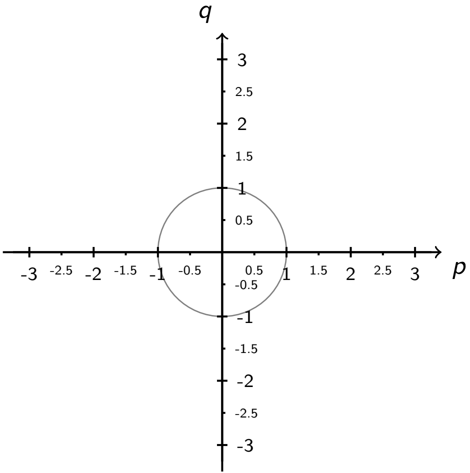

The unit circle (C) stands for transformations with $s = 1$, i.e., for $\mathcal{F}_{\mathtt{r}}$.

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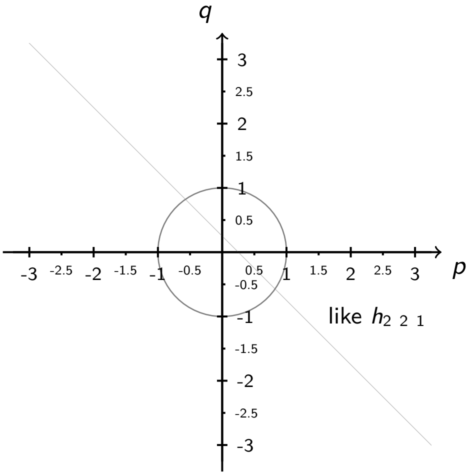

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$h_{ijk} : ip + jq = k - 0.5$

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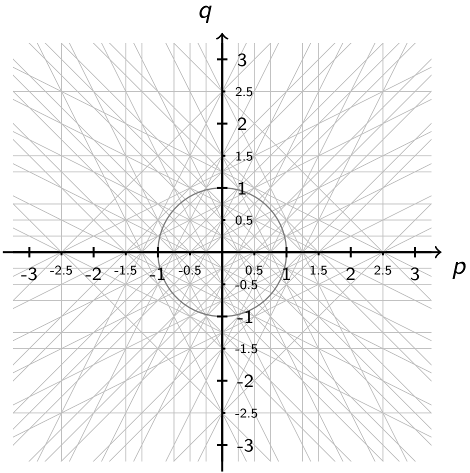

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and for
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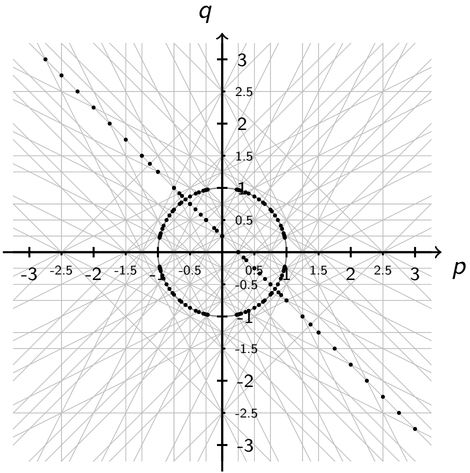

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cutting $\mathbb{R}^2$ and C into sets $\mathcal{A}(m,n)$ and $\mathcal{A}_C(m,n)$ of

- points,
- line segments,
- convex regions.

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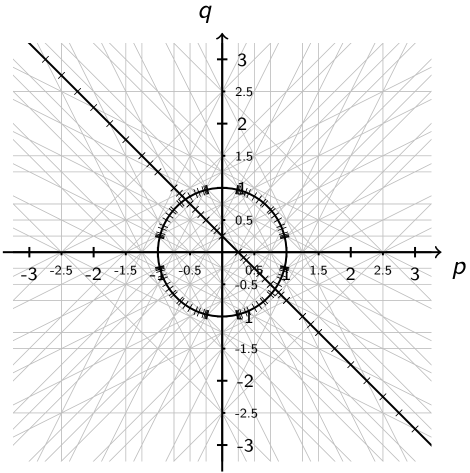

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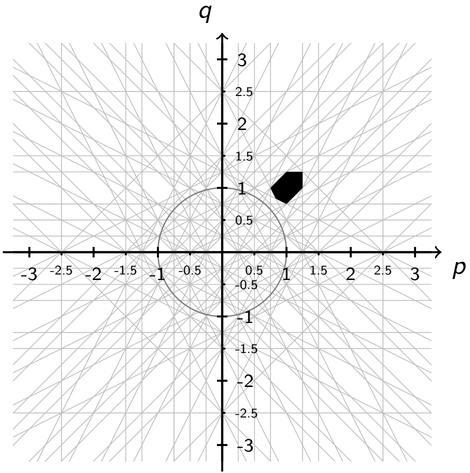

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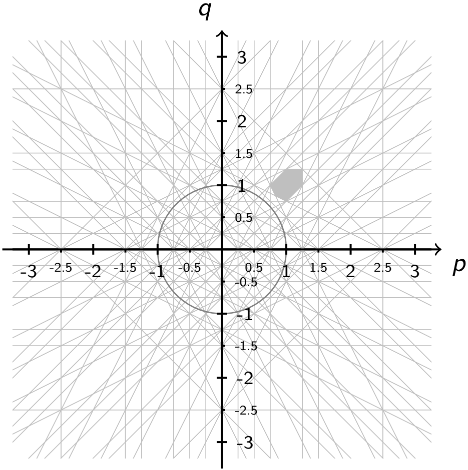

Theorem

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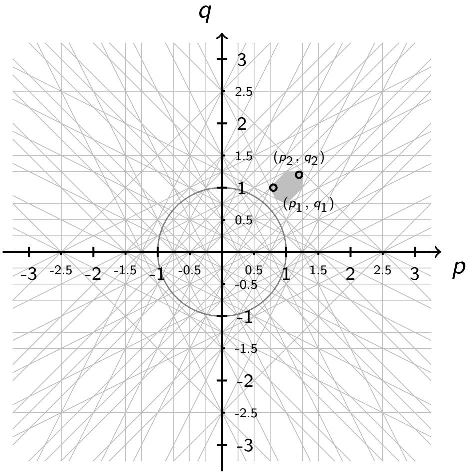

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Let $\varphi \in \mathcal{A}(m, n)$ *and* $(p_1, q_1), (p_2, q_2) \in \varphi$

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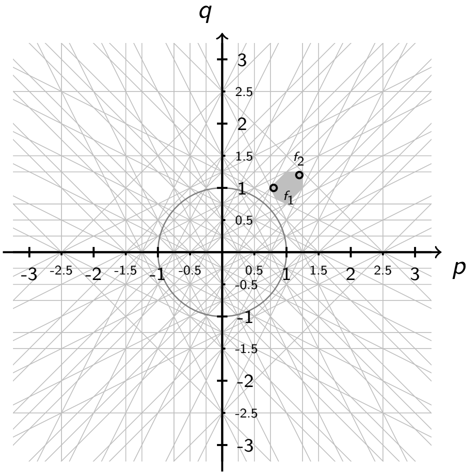

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Then $f_1(A) = f_2(A)$.

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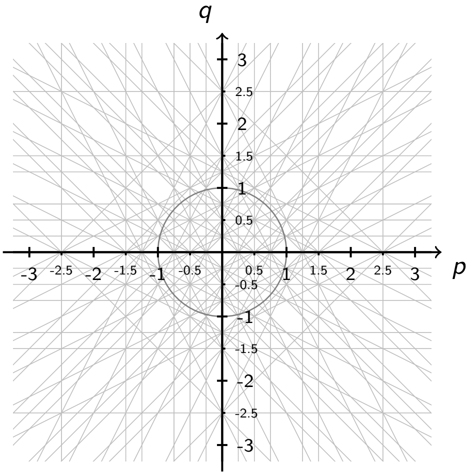

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Then $f_1(A) = f_2(A)$.

The same holds for $\mathcal{A}_C(m, n)$.

Our Characterization

Theorem

If $m \approx n$ *then*

1. $|\mathcal{A}_C(m, n)| \in \Theta(n^3)$,
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Our Image Matching Algorithm

Input: Images A of size m and B of size n

Implicit: Transformation class $\mathcal{F}_{\mathtt{sr}}$

Output: $f \in \mathcal{F}_{\mathtt{sr}}$ with minimum $\Delta = (f(A), B)$

1.
2.
3.
4.
5.
6.

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	overall time	$O(n^6)$

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Our Image Matching Algorithm

Input: Images A of size m and B of size n

Implicit: Transformation class $\mathcal{F}_{\mathtt{r}}$

Output: $A' = f(A)$ with $f \in \mathcal{F}_{\mathtt{r}}$ and minimum $\Delta = (f(A), B)$

1.	construct $\mathcal{A}_C(m,n)$	$O(n^3 \log n)$ (MergeSort)[2]
2.	traverse all faces φ of $\mathcal{A}_C(m,n)$	$O(n^3)$ (Linear Traversal)
3.	compute $f(A)$ for given φ	$O(1)$ (amortized)
4.	obtain $\Delta = (f(A), B)$	$O(1)$ (amortized)
5.	return A'	$O(1)$
	overall time	$O(n^3 \log n)$

[2] We assume $m \approx n$.

Recently raised Questions

1. How to resolve the $\log n$-discrepancy between $|\mathcal{A}_C(m,n)|$ and the $\mathcal{F}_{\mathtt{r}}$-algorithm's preprocessing time?
2. What is the "exact" cardinality of $\mathcal{A}(m,n)$?

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2. What is the "exact" cardinality of $\mathcal{A}(m, n)$?
   Same question: How fast does our algorithm run for $\mathcal{F}_{\mathtt{sr}}$?

New Results, partial Answers

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If $m \approx n$ then

1. *We can preprocess the data structure for $\mathcal{A}_C(m, n)$ in $O(n^3)$ and*
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Preprocessing $\mathcal{A}_C(m, n)$ efficiently

Basic Ideas:

1. Points and line segements are given by line intersections with C.
2. So, we will compute the intersections

$$p = \frac{i(k-0.5) \pm j\sqrt{i^2+j^2-(k-0.5)^2}}{i^2+j^2}$$

$$q = \frac{j(k-0.5) \mp i\sqrt{i^2+j^2-(k-0.5)^2}}{i^2+j^2}$$

   and sort them according to their linear occurence on C.
3. But we will now use RadixSort instead of MergeSort to get linear preprocessing time.
4. Problem: All (p, q) are irrational.
5. Hence, we have to show that $O(\log n)$ precision is enough.
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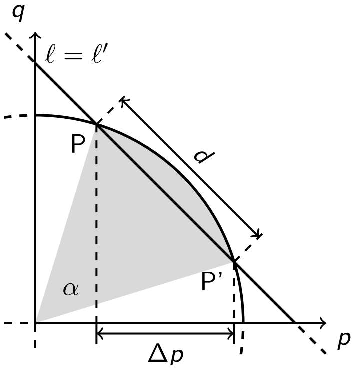


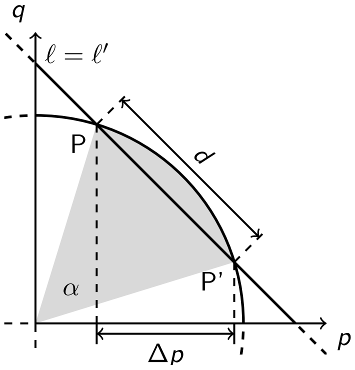
- $i = i'$, $j = j'$, $k = k'$.
- Since $4i^2 + 4j^2 \geq (2k-1)^2$:

$$d = \|P - P'\|$$
$$= \sqrt{\frac{4i^2 + 4j^2 - (2k-1)^2}{i^2 + j^2}}$$
$$\geq \sqrt{\frac{1}{2n^2}} \geq \frac{1}{\sqrt{2}n}.$$

- Hence, d is huge compared to $\Omega(n^{-5})$.

The $O(\log n)$ Precision Bound

What ist the minimum distance between the intersections of C with two lines ℓ and ℓ'?



- $i = i'$, $j = j'$, $k = k'$.
- Since $4i^2 + 4j^2 \geq (2k-1)^2$:

$$d = \|P - P'\| = \sqrt{\frac{4i^2 + 4j^2 - (2k-1)^2}{i^2 + j^2}} \geq \sqrt{\frac{1}{2n^2}} \geq \frac{1}{\sqrt{2}n}.$$

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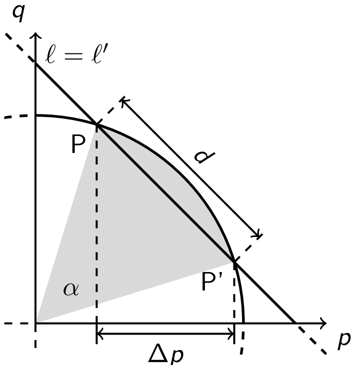

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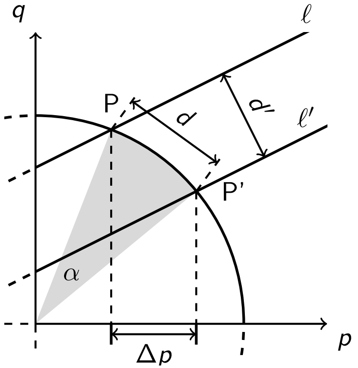

- Point on ℓ': $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2k'-1}{2j'} \end{pmatrix}$.
- Then

$$d \geq d' = \frac{|ip_0 + jq_0 - (k - 0.5)|}{\sqrt{i^2 + j^2}}$$

$$= \left| \frac{j(2k' - 1) - j'(2k - 1)}{2j'\sqrt{i^2 + j^2}} \right|$$

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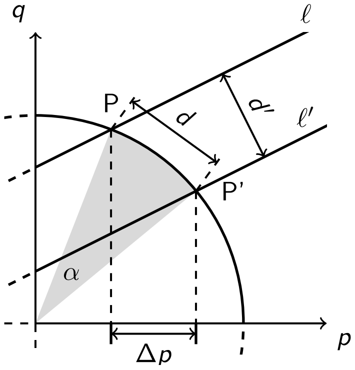

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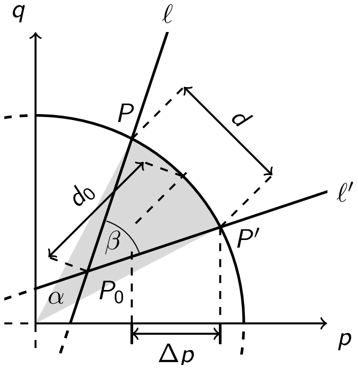

- d is found by distance from P_0 to C and angle β between ℓ and ℓ'.
- We show $d_0 \geq \frac{1}{34n^4}$ and $\beta \geq \frac{1}{n}$.
- $\beta \geq \frac{\pi}{2}$ leads to $d \geq \Omega(n^{-4})$.
- If $\beta < \frac{\pi}{2}$ then isosceles triangle is worst case:

$$d \geq d_0 \tan \frac{\beta}{2} \geq \frac{1}{34n^4} \tan \frac{1}{2n} \geq \frac{1}{68n^5}$$

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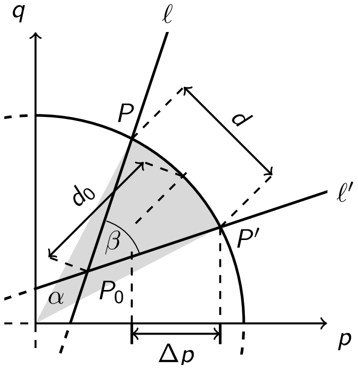

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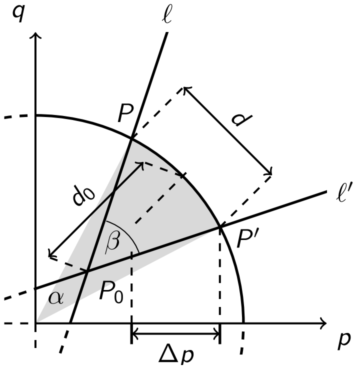

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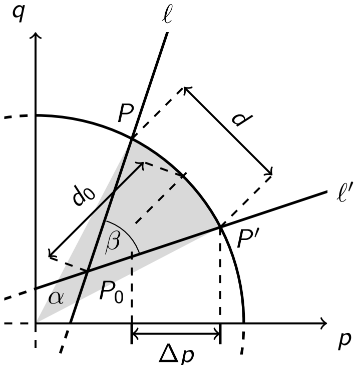

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What ist the minimum distance between the intersections of C with two lines ℓ and ℓ'?

- The lower bound on d gives that $\alpha \geq \frac{1}{68n^5}$.
- Now Δp can be estimated:

$$\Delta p = 1 - \cos\alpha \geq 1 - \cos\left(\frac{1}{68n^5}\right) \geq 2^{-14}n^{-10}.$$

- The order on p-coordinates gives the order of intersection on C.
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The Cardinality of $\mathcal{A}(m, n)$

Basic Ideas:

1. #points ≈ #line segments ≈ #regions
2. So, we will count line segments then.
3. All lines $h_{ijk} : ip + jq = k - 0.5$ are cut into segments by other lines.
4. Hence, count the number of intersection points on h_{ijk} to get the number of its segments:

$$p = \frac{j'(k - 0.5) - j(k' - 0.5)}{ij' - i'j} \quad \text{and} \quad q = \frac{i(k' - 0.5) - i'(k - 0.5)}{ij' - i'j}$$

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Make it easier:

- $H_a(n) = \{(i, j, k) \mid 0.5n \leq i, j \leq n, 1 \leq k \leq 0.01n, \gcd(i, j) = 1\}$
- $H_b(n) = \{(i, j, k) \mid -n \leq i, j \leq n, 1 \leq k \leq 0.01n\}$

Consider only segments obtained by cutting lines $h_{i(-j)k}$ with $h_{i'j'k'}$ where $(i, j, k) \in H_a(n)$ and $(i', j', k') \in H_b(n)$.

The Cardinality of $\mathcal{A}(m, n)$

This means: Count for each $(i, j, k) \in H_a(n)$ the number of $(i', j', k') \in H_b(n)$, which give a new fraction:

$$p = \frac{N}{D} = \frac{j'(k - 0.5) + j(k' - 0.5)}{ij' + i'j}$$

1. We have i and j coprime und thus use Bézout's Lemma.
2. Then D can be any number in $\{1, \ldots, j \cdot n\}$ by the choice of i', j'.
3. Particularly, any of the $\Omega(n^2/\log n)$ primes in $\{0.04n^2, \ldots, 0.5n^2\}$.
4. By k' we can still choose $0.01n$ many numerators with $N < 0.04n^2$.
5. That gives $\Omega(n^3/\log n)$ segments on $h_{i(-j)k}$.
6. Since $|H_a(n)| \in \Omega(n^3)$ we get $\Omega(n^6/\log n)$ segments.

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End of Talk

Thank you for your attention!