Periodic string comparison

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Introduction

String matching: finding an exact pattern in a string

String comparison: finding similar patterns in two strings

(Also known as "approximate string matching", no relation to approximation algorithms!)

Applications: computational biology, image recognition, ...

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Standard types of string comparison:

- global: whole string vs whole string
- local: substrings vs substrings

Main focus of this work:

- semi-local: whole string vs substrings; prefixes vs suffixes

Main tool: implicit unit-Monge matrices

Integers: $\ldots -2, -1, 0, 1, 2, \ldots$

Odd half-integers: $\ldots -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

We consider finite and infinite integer matrices over integer and odd half-integer indices. For simplicity, index range will usually be ignored.

A permutation matrix is a $0/1$ matrix with exactly one nonzero per row and per column

Given matrix D, its distribution matrix is $D^{\Sigma}(i,j) = \sum_{i'>i, j'<j} D(i',j')$

In other words, $D^{\Sigma}(i,j)$ is the sum of all $D(i',j')$, where (i',j') is dominated by (i,j)

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Given matrix E, its density matrix is

$$E^{\square}(i,j) = E(i^-,j^+) - E(i^-,j^-) - E(i^+,j^+) + E(i^+,j^-)$$

where $i^{\pm} = i \pm \frac{1}{2}$; D^{Σ}, E over integers; D, $E^{\square}$ over odd half-integers

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$(D^{\Sigma})^{\square} = D$ for all D

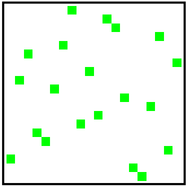
Matrix E is simple, if $(E^{\square})^{\Sigma} = E$

Matrix E is Monge, if $E^{\square}$ is nonnegative

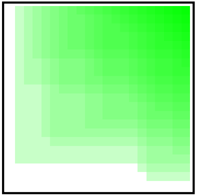
Intuition: border-to-border distances in a (weighted) planar graph

Matrix E is unit-Monge, if $E^{\square}$ is a permutation matrix

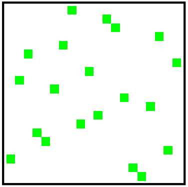
Intuition: border-to-border distances in a grid-like graph



P



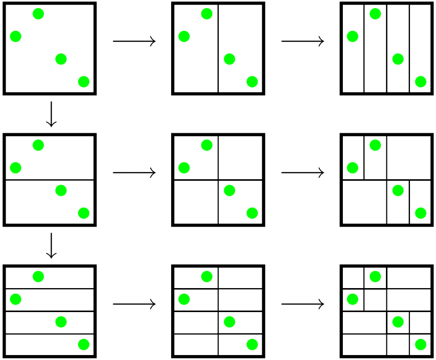
P^{Σ}



$(P^{\Sigma})^{\square} = P$

Implicit P^{Σ}: range tree on nonzeros of P [Bentley: 1980]

- binary search tree by i-coordinate
- under every node, binary search tree by j-coordinate



Implicit P^{Σ} (contd.)

Every node of the range tree represents a canonical range (rectangular region), and stores its nonzero count

Overall, $\leq n \log n$ canonical ranges are non-empty

Range tree supports dominance counting queries: how many nonzeros are dominated by a given point? Answered by decomposing query range into $\leq \log^2 n$ disjoint canonical ranges.

Total size $O(n \log n)$, query time $O(\log^2 n)$

There are asymptotically more efficient (but less practical) data structures

Matrix $\odot$-multiplication (a.k.a. distance, $(\min, +)$ or tropical multiplication)

$A \odot B = C \qquad C(i,k) = \min_j\bigl(A(i,j) + B(j,k)\bigr)$

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$A \odot B = C \qquad C(i,k) = \min_j \bigl(A(i,j) + B(j,k)\bigr)$

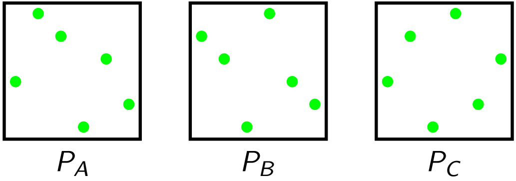
Matrix classes closed under $\odot$-multiplication:

- general numerical (integer, real) matrices
- Monge matrices
- simple unit-Monge matrices

Simple unit-Monge matrices of size n form an aperiodic monoid (i.e. a monoid as far as possible from a group) under $\odot$-multiplication

We call it the seaweed monoid $\mathcal{T}_n$

Simple unit-Monge matrices: $\odot$-multiplication = seaweed composition



$$P_A^\Sigma \odot P_B^\Sigma = P_C^\Sigma$$

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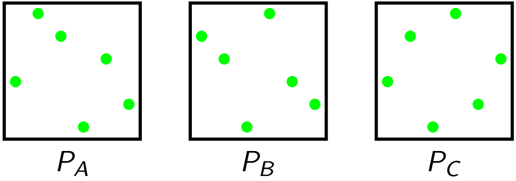
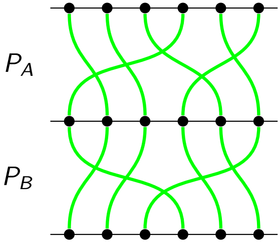
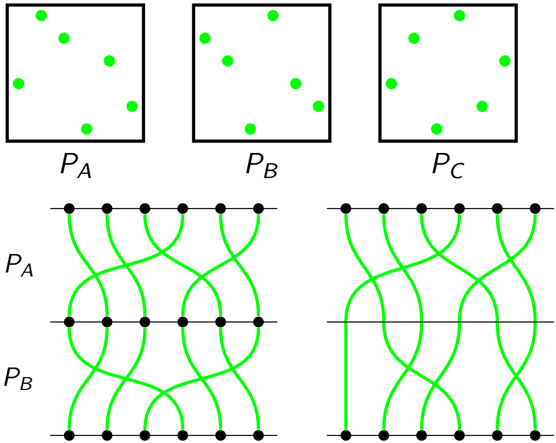




Simple unit-Monge matrices: $\odot$-multiplication = seaweed composition

$$P_A^{\Sigma} \odot P_B^{\Sigma} = P_C^{\Sigma}$$



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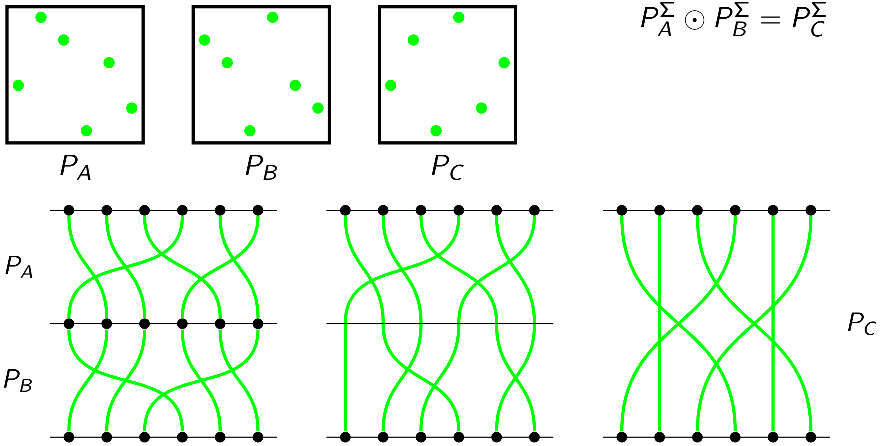

Seaweeds: similar to braids, generated by strand crossings

Unlike in braids, all seaweed crossings are

- level (not underpass/overpass)
- idempotent, i.e. two seaweeds can cross at most once

Seaweed composition: associative, no inverse (a crossing cannot be cancelled)

Identity: $1 \odot x = x$, no seaweeds crossing

Zero: $0 \odot x = 0$, all seaweeds crossing

$$1 = \begin{pmatrix} \bullet & \cdot & \cdot & \cdot & \cdot \\ \cdot & \bullet & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bullet & \cdot & \cdot \\ \cdot & \cdot & \cdot & \bullet & \cdot \\ \cdot & \cdot & \cdot & \cdot & \bullet \end{pmatrix}^{\Sigma} \qquad 0 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \bullet \\ \cdot & \cdot & \cdot & \bullet & \cdot \\ \cdot & \cdot & \bullet & \cdot & \cdot \\ \cdot & \bullet & \cdot & \cdot & \cdot \\ \bullet & \cdot & \cdot & \cdot & \cdot \end{pmatrix}^{\Sigma}$$

The seaweed monoid $\mathcal{T}_n$:

- $n!$ elements (permutations of size n)
- $n-1$ generators $g_1, g_2, \ldots, g_{n-1}$ (elementary crossings)

$g_i^2 = g_i$ for all i (idempotence)

$g_i g_j = g_j g_i$ $j - i > 1$ (far commutativity)

$g_i g_j g_i = g_j g_i g_j$ $j - i = 1$ (braid relations)

Computation: a confluent rewriting system can be obtained by software (Semigroupe, GAP)

Generalisation: Coxeter monoids (subgroup monoids in groups)

[Tsaranov: 90]

The seaweed monoid $\mathcal{T}_3$

Generators: 1, $a = g_1$, $b = g_2$

Other elements: ab, ba, $aba = 0$

Rewriting system:

$aa \longrightarrow a$
$bb \longrightarrow b$
$bab \longrightarrow 0$
$aba \longrightarrow 0$

The seaweed monoid $\mathcal{T}_4$

Generators: 1, $a = g_1$, $b = g_2$, $c = g_3$

Other elements: ab, ac, ba, bc, cb, aba, abc, acb, bac, bcb, cba, $abac$, $abcb$, $acba$, $bacb$, $bcba$, $abacb$, $abcba$, $bacba$, $abacba = 0$

Rewriting system:

$aa \longrightarrow a$
$bb \longrightarrow b$
$ca \longrightarrow ac$
$cc \longrightarrow c$
$bab \longrightarrow aba$
$cbc \longrightarrow bcb$
$cbac \longrightarrow bcba$
$abacba \longrightarrow 0$

The implicit matrix $\odot$-multiplication problem

Given permutation matrices P_A, P_B, compute P_C, such that $P_A^\Sigma \odot P_B^\Sigma = P_C^\Sigma$

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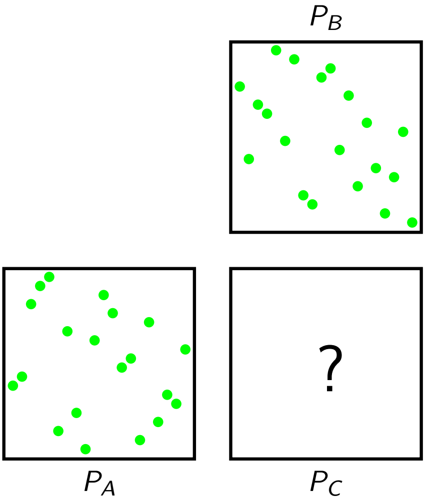
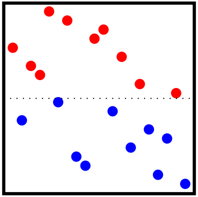
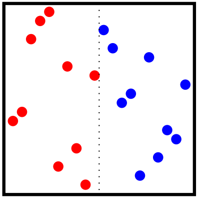
Matrix $\odot$-multiplication: running time

matrix type	time	
general	$O(n^3)$	standard
Monge	$O(n^2)$	by [Aggarwal+: 1987]
implicit simple unit-Monge (P^Σ)		
	$O(n^{1.5})$	[T: 2006]
	$O(n \log n)$	[T: NEW]


P_B
?
P_A
P_C

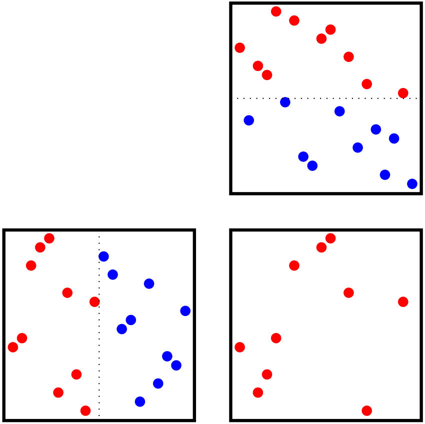
$P_{B,lo},\ P_{B,hi}$





$P_{A,lo},\ P_{A,hi}$

$P_{B,lo}$, $P_{B,hi}$



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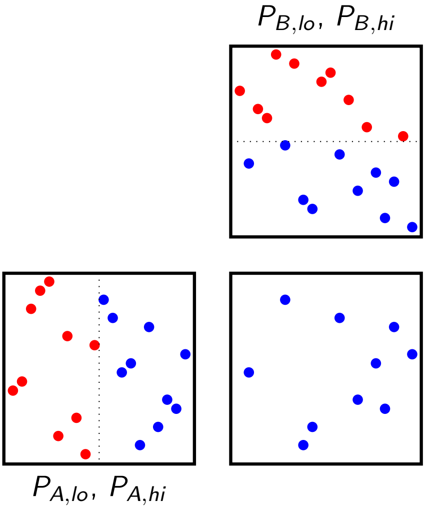
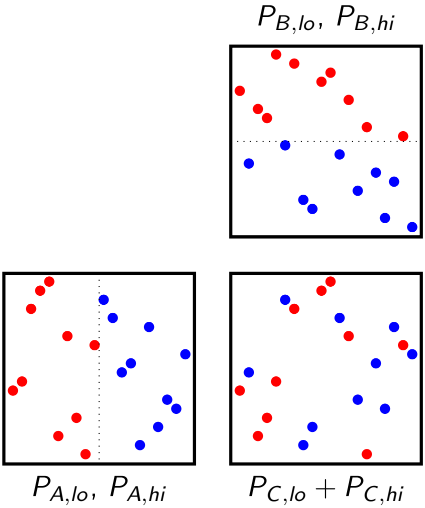


$P_{B,lo}$, $P_{B,hi}$
$P_{A,lo}$, $P_{A,hi}$


$P_{B,lo}$, $P_{B,hi}$
$P_{A,lo}$, $P_{A,hi}$
$P_{C,lo} + P_{C,hi}$

Implicit matrix $\odot$-multiplication: the algorithm

$P^{\Sigma}_{C}(i,k) = \min_j \left(P^{\Sigma}_{A}(i,j) + P^{\Sigma}_{B}(j,k)\right)$

Divide-and-conquer on the range of j

Divide P_A horizontally, P_B vertically; two subproblems of effective size $n/2$:

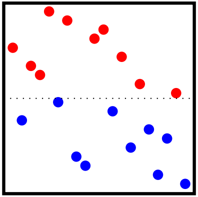
$P^{\Sigma}_{A,lo} \odot P^{\Sigma}_{B,lo} = P^{\Sigma}_{C,lo} \qquad P^{\Sigma}_{A,hi} \odot P^{\Sigma}_{B,hi} = P^{\Sigma}_{C,hi}$

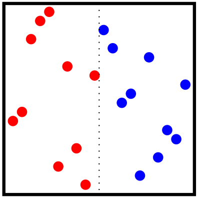
Conquer: most (but not all!) nonzeros of $P_{C,lo}$, $P_{C,hi}$ appear in P_C

Missing nonzeros can be obtained in time $O(n)$ using the Monge property

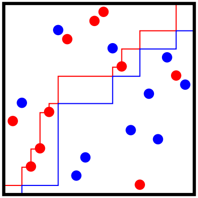
Overall time $O(n \log n)$

$P_{B,lo}, P_{B,hi}$

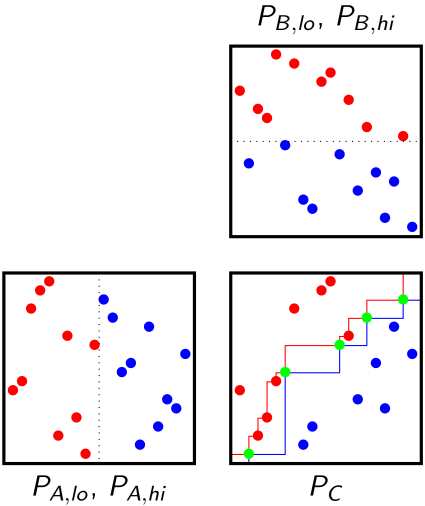




$P_{A,lo}, P_{A,hi}$



$P_{C,lo} + P_{C,hi}$



$P_{B,lo}, P_{B,hi}$
$P_{A,lo}, P_{A,hi}$
P_C

Consider strings (= sequences) over an alphabet of size σ

Distinguish contiguous substrings and not necessarily contiguous subsequences

Special cases of substring: prefix, suffix

Notation: strings a, b of length m, n respectively

Assume where necessary: $m \leq n$; m, n reasonably close

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The longest common subsequence (LCS) score:

- length of longest string that is a subsequence of both a and b
- equivalently, alignment score, where $score(match) = 1$ and $score(mismatch) = 0$

The LCS problem

Give the LCS score for a vs b

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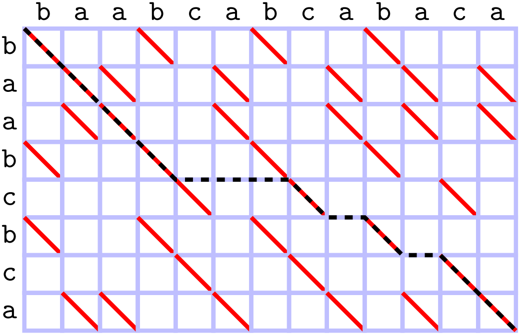
Give the LCS score for a vs b

LCS: running time

$O(mn)$		[Wagner, Fischer: 1974]
$O\left(\frac{mn}{\log n}\right)$	$\sigma = O(1)$	[Masek, Paterson: 1980]
		[Crochemore+: 2003]
$O\left(\frac{mn(\log \log n)^2}{\log n}\right)$		[Paterson, Dancik: 1994]

LCS on the alignment graph (directed, acyclic)



blue = 0
red = 1

LCS("baabcbca", "baabcabcabaca") = "baabcbca"

LCS = highest-score corner-to-corner path

LCS: dynamic programming (DP) algorithm [Wagner, Fischer: 1974]

Sweep alignment graph, respecting node dependencies

Running time $O(mn)$

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Sweep alignment graph, respecting node dependencies

Running time $O(mn)$

LCS: micro-block DP algorithm [Masek, Paterson: 1980]

Sweep alignment graph in square blocks, respecting block dependencies

Block size: $t = O(\log n)$

Block interface: $O(t)$ inputs/outputs, each of size $O(\log \sigma)$

Use precomputed mapping of all possible input/output combinations

Running time $O\left(\frac{mn}{\log n}\right)$ when $\sigma = O(1)$, even on log-cost RAM

The semi-local LCS problem

Give the (implicit) matrix of $O(m^2 + n^2)$ LCS scores:

- string-substring LCS: string a vs every substring of b
- prefix-suffix LCS: every prefix of a vs every suffix of b
- symmetrically, substring-string and suffix-prefix LCS

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The three-way semi-local LCS problem

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Suitable for $m \gg n$

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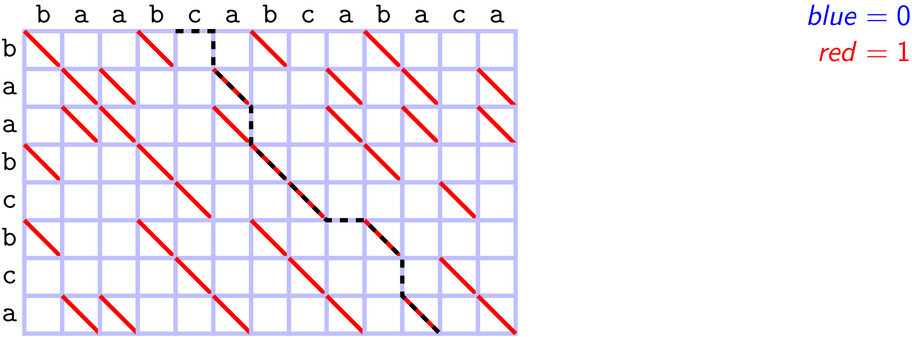
Suitable for $m \gg n$

Cf.: dynamic programming gives prefix-prefix LCS

Semi-local LCS on the alignment graph



$LCS(\texttt{"baabcbca"}, \texttt{"...cabcaba..."}) = \texttt{"abcba"}$

Semi-local LCS = all highest-score border-to-border paths
(string-substring = top-to-bottom, etc.)

The LCS problem is a special case of the alignment score problem with weighted matches, mismatches and gaps

- LCS score: $w_{match} = 1$, $w_{mismatch} = w_{gap} = 0$
- Levenshtein score: $w_{match} = 2$, $w_{mismatch} = 1$, $w_{gap} = 0$

An alignment score is rational, if w_{match}, $w_{mismatch}$, w_{gap} are rational; reduces to LCS score by constant-factor blow-up of alignment graph

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The semi-local alignment score problem: string-substring, prefix-suffix, substring-string, suffix-prefix alignment scores

Edit distance: minimum cost to transform a into b by weighted character edits (insertion, deletion, substitution)

The semi-local edit distance problem: semi-local alignment score problem with $w_{match} = 0$, $w_{mismatch} = -w_{sub}$, $w_{gap} = -w_{indel}$

The semi-local LCS score matrix

0	1	2	3	4	5	6	6	7	8	8	8	8	8
−1	0	1	2	3	4	5	5	6	7	7	7	7	7
−2	−1	0	1	2	3	4	4	5	6	6	6	6	7
−3	−2	−1	0	1	2	3	3	4	5	5	6	6	7
−4	−3	−2	−1	0	1	2	2	3	4	4	(5)	5	6
−5	−4	−3	−2	−1	0	1	2	3	4	4	5	5	6
−6	−5	−4	−3	−2	−1	0	1	2	3	3	4	4	5
−7	−6	−5	−4	−3	−2	−1	0	1	2	2	3	3	4
−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	3	4
−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4
−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3
−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2
−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1
−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0

$a =$ "baabcbca"

$b =$ "baabcabcabaca"

$A(0, 13) = LCS(a, b) = 8$

$b' =$ "...cabcaba..."

$A(4, 11) = LCS(a, b') = 5$

$A(i, j) = j - i$ if $i > j$

Semi-local LCS: output representation and running time

size	query time		
$O(n^2)$	$O(1)$		trivial
$O(m^{1/2}n)$	$O(\log n)$	string-substring	[Alves+: 2003]
$O(n)$	$O(n)$	string-substring	[Alves+: 2005]
$O(n \log n)$	$O(\log^2 n)$		[T: 2006]

running time		
$O(mn^2)$		naive
$O(mn)$	string-substring	[Schmidt: 1998]
	string-substring	[Alves+: 2005]
$O(mn)$		[T: 2006]
$O\left(\frac{mn}{\log^{0.5} n}\right)$		[T: 2006]
$O\left(\frac{mn(\log \log n)^2}{\log n}\right)$		[T: 2007]

A: the semi-local LCS score matrix for a vs b

$A(i, j)$: the number of matched characters for a vs substring of b

$Q(i, j) = j - i - A(i, j)$: the number of unmatched characters

Properties of matrix Q:

- Q is simple unit-Monge
- therefore, $Q = P^{\Sigma}$ for some permutation matrix P

$P = Q^{\square} = -A^{\square}$ is an implicit representation of A

Range tree for P: memory $O(n \log n)$, query time $O(\log^2 n)$

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0	1	2	3	4	5	6	6	7	8	8	8	8	8
−1	0	1	2	3	4	5	5	6	7	7	7	7	7
−2	−1	0	1	2	3	4	4	5	6	6	6	6	7
−3	−2	−1	0	1	2	3	3	4	5	5	6	6	7
−4	−3	−2	−1	0	1	2	2	3	4	4	5	5	6
−5	−4	−3	−2	−1	0	1	2	3	4	4	5	5	6
−6	−5	−4	−3	−2	−1	0	1	2	3	3	4	4	5
−7	−6	−5	−4	−3	−2	−1	0	1	2	2	3	3	4
−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	3	4
−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4
−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3
−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2
−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1
−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0

$a =$ "baabcbca"

$b =$ "baabcabcabaca"

$b' =$ "...cabcaba..."

$A(4, 11) = LCS(a, b') = 5$

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The semi-local LCS score matrix

0	1	2	3	4	5	6	6	7	8	8	8	8	8
−1	0	1	2	3	4	5	5	6	7	7	7	7	7
−2	−1	0	1	2	3	4	4	5	6	6	6	6	7
−3	−2	−1	0	1	2	3	3	4	5	5	6	6	7
−4	−3	−2	−1	0	1	2	2	3	4	4	(5)	5	6
−5	−4	−3	−2	−1	0	1	2	3	4	4	5	5	6
−6	−5	−4	−3	−2	−1	0	1	2	3	3	4	4	5
−7	−6	−5	−4	−3	−2	−1	0	1	2	2	3	3	4
−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	3	4
−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4
−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3
−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2
−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1
−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0

$a =$ "baabcbca"

$b =$ "baabcabcabaca"

$b' =$ "...cabcaba..."

$A(4, 11) = LCS(a, b') = 5$

$A(i, j) = j - i$ if $i > j$

blue: difference 0

red: difference 1

Semi-local string comparison

Highest-score matrices

The semi-local LCS score matrix

0	1	2	3	4	5	6	6	7	8	8	8	8	8
−1	0	1	2	3	4	5	5	6	7	7	7	7	7
−2	−1	0	1	2	3	4	4	5	6	6	6	6	7
−3	−2	−1	0	1	2	3	3	4	5	5	6	6	7
−4	−3	−2	−1	0	1	2	2	3	4	4	(5)	5	6
−5	−4	−3	−2	−1	0	1	2	3	4	4	5	5	6
−6	−5	−4	−3	−2	−1	0	1	2	3	3	4	4	5
−7	−6	−5	−4	−3	−2	−1	0	1	2	2	3	3	4
−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	3	4
−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4
−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3
−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2
−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1
−13	−12	−11	−10	−9	−8	−7	−6	−5	−4	−3	−2	−1	0

$a =$ `"baabcbca"`

$b =$ `"baabcabcabaca"`

$b' =$ `"...cabcaba..."`

$A(4, 11) = LCS(a, b') = 5$

$A(i, j) = j - i$ if $i > j$

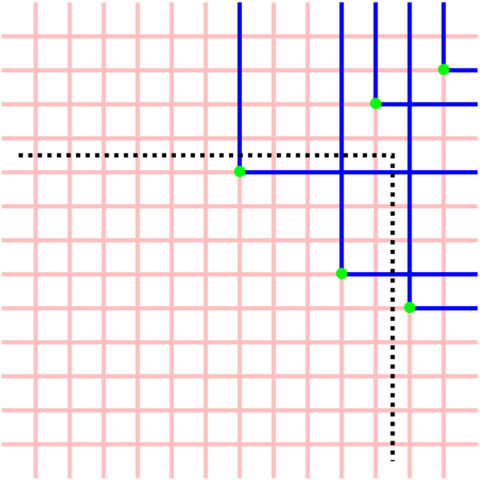
blue: difference 0

red: difference 1

green: $P(i, j) = 1$

$A(i, j) = j - i - P^{\Sigma}(i, j)$

The semi-local LCS score matrix



$a =$ "baabcbca"

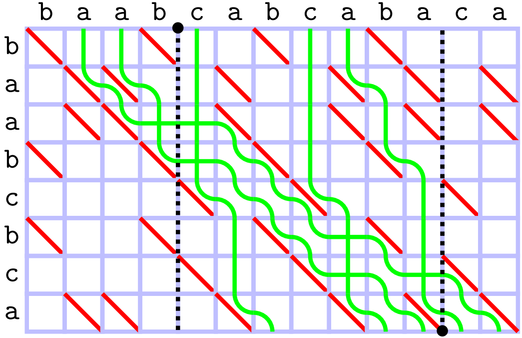
$b =$ "baabcabcabaca"

$b' =$ "...cabcaba..."

$A(4, 11) = LCS(a, b') =$
$11 - 4 - P^{\Sigma}(i, j) =$
$11 - 4 - 2 = 5$

P gives an implicit representation of A

The seaweeds in the alignment graph



$a =$ "baabcbca"

$b =$ "baabcabcabaca"

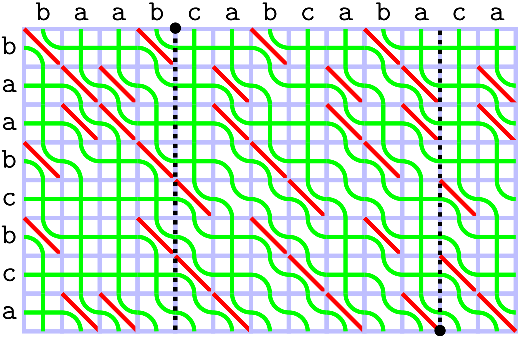
$b' =$ "...cabcaba..."

$A(4, 11) = LCS(a, b') =$
$11 - 4 - P^{\Sigma}(i, j) =$
$11 - 4 - 2 = 5$

P gives an implicit representation of A

$P(i, j) = 1$ corresponds to seaweed $(top, i) \rightsquigarrow (bottom, j)$

The seaweeds in the alignment graph



$a =$ "baabcbca"

$b =$ "baabcabcabaca"

$b' =$ "...cabcaba..."

$A(4, 11) = LCS(a, b') =$
$11 - 4 - P^{\Sigma}(i, j) =$
$11 - 4 - 2 = 5$

P gives an implicit representation of A

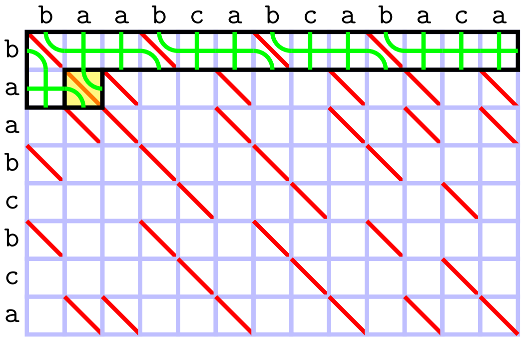
$P(i, j) = 1$ corresponds to seaweed $(top, i) \rightsquigarrow (bottom, j)$

Also define $top \rightsquigarrow right$, $left \rightsquigarrow right$, $left \rightsquigarrow bottom$ seaweeds

Gives complete border-to-border graph-theoretic matching

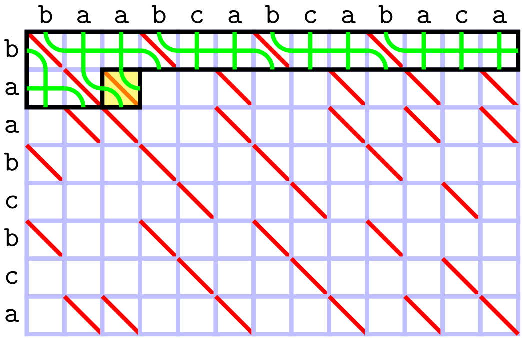
The seaweed algorithm

The seaweed algorithm



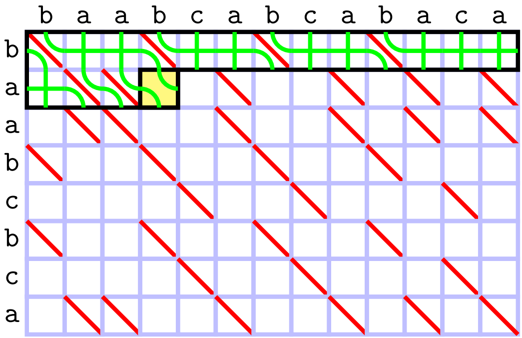
The seaweed algorithm

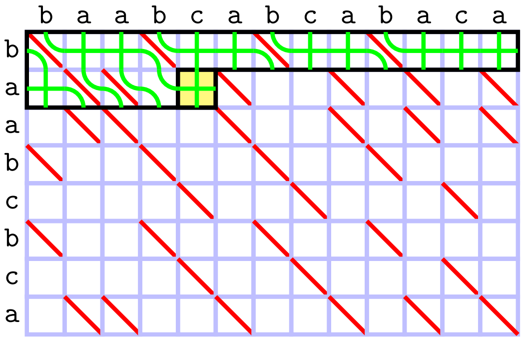
The seaweed algorithm



The seaweed algorithm

The seaweed algorithm




b a a b c a b c a b a c a
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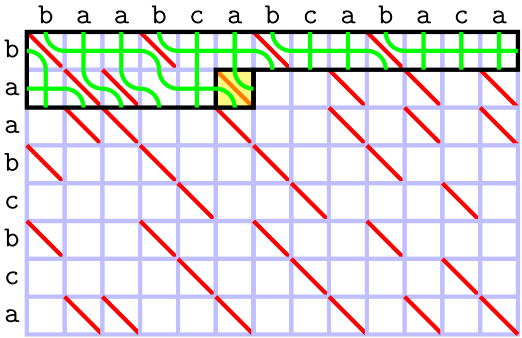
c
a

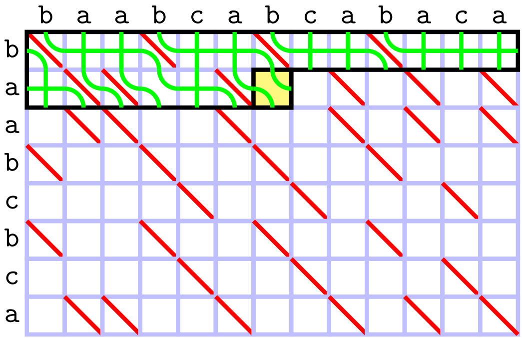
The seaweed algorithm

The seaweed algorithm



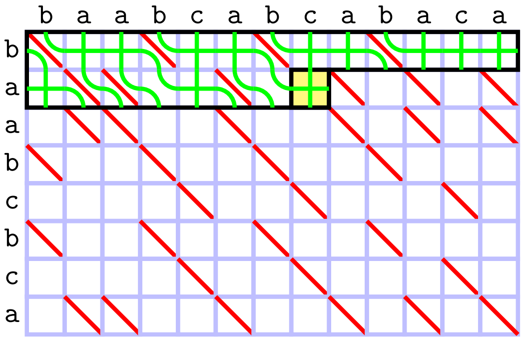
The seaweed algorithm

The seaweed algorithm



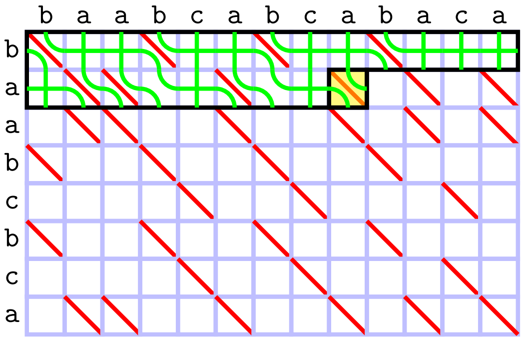
The seaweed algorithm

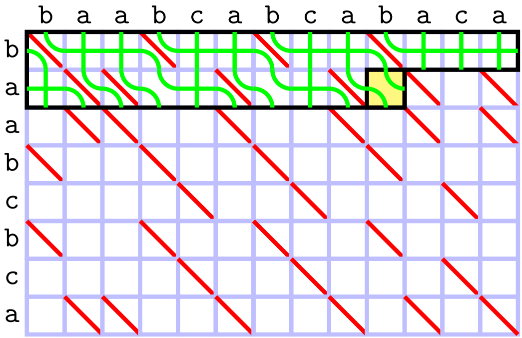
The seaweed algorithm



The seaweed algorithm

The seaweed algorithm




b a a b c a b c a b a c a
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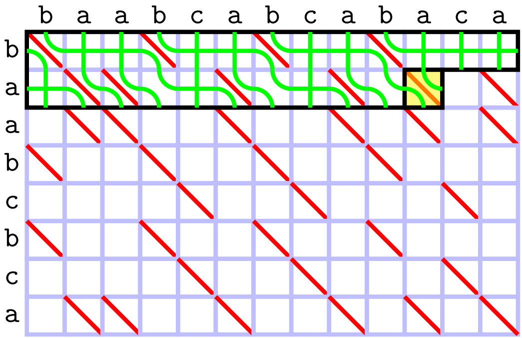
a

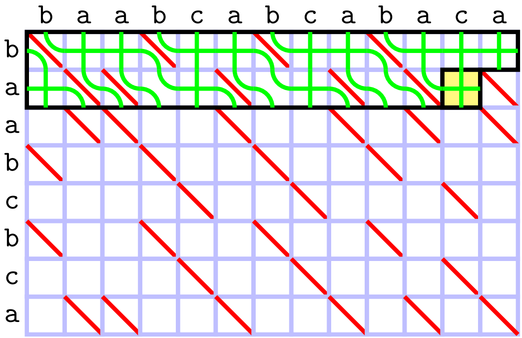
The seaweed algorithm

The seaweed algorithm



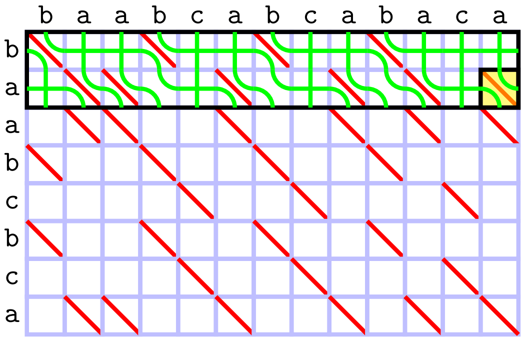
The seaweed algorithm

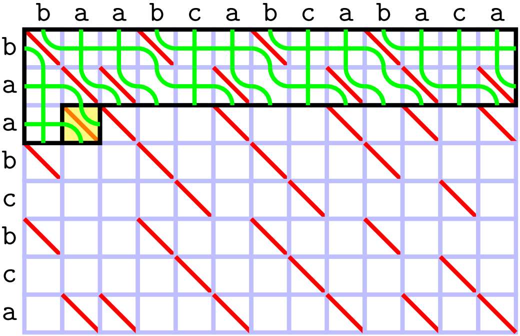
The seaweed algorithm



The seaweed algorithm

The seaweed algorithm





b a a b c a b c a b a c a
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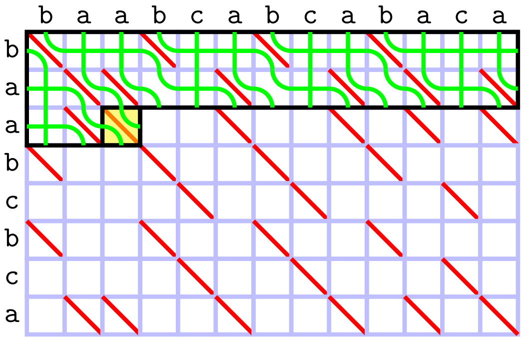
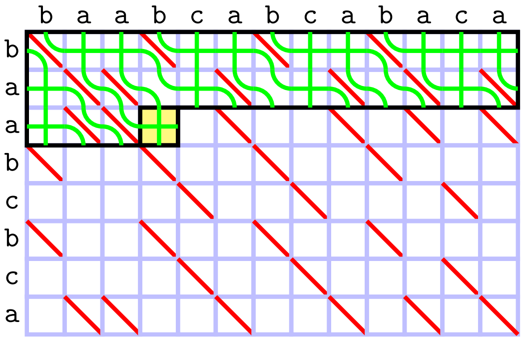
The seaweed algorithm

The seaweed algorithm



The seaweed algorithm

The seaweed algorithm



The seaweed algorithm

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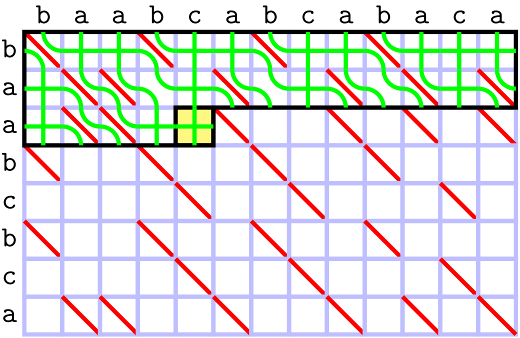

The seaweed algorithm

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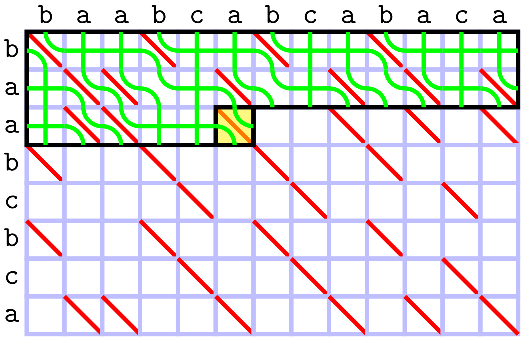
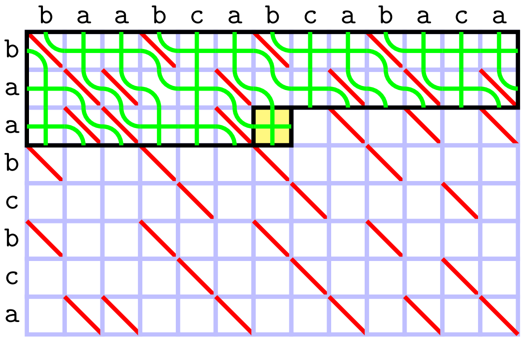


The seaweed algorithm

The seaweed algorithm



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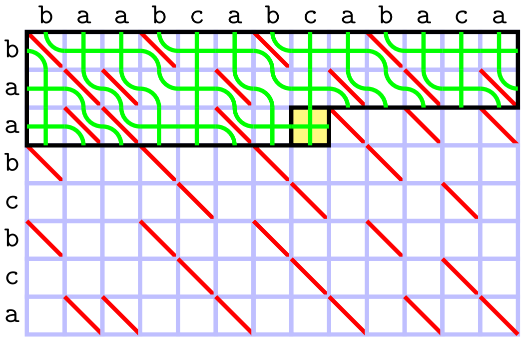

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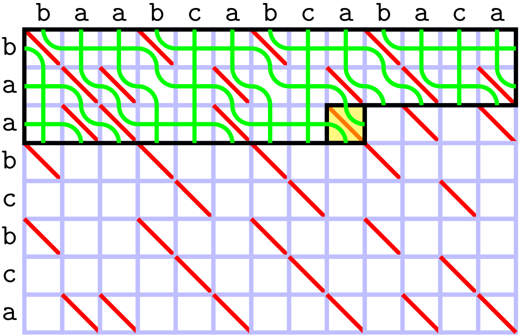
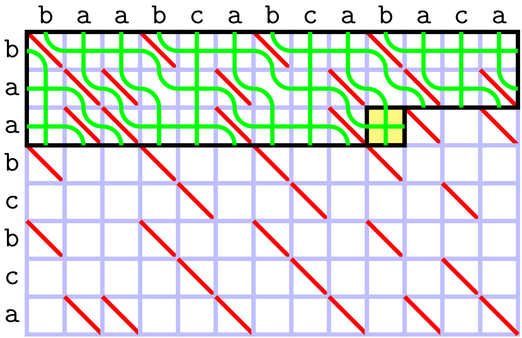


The seaweed algorithm

The seaweed algorithm



The seaweed algorithm

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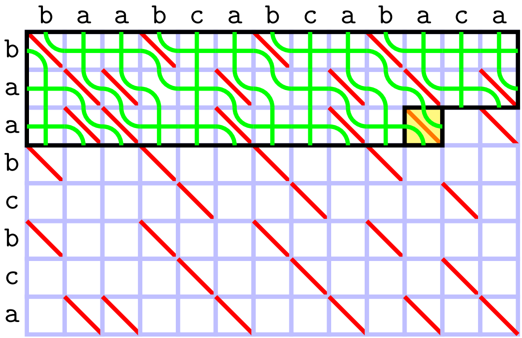
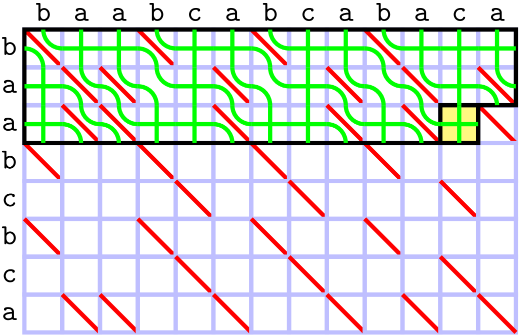


The seaweed algorithm

The seaweed algorithm



The seaweed algorithm

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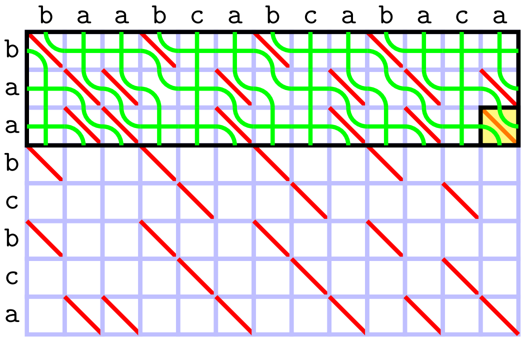

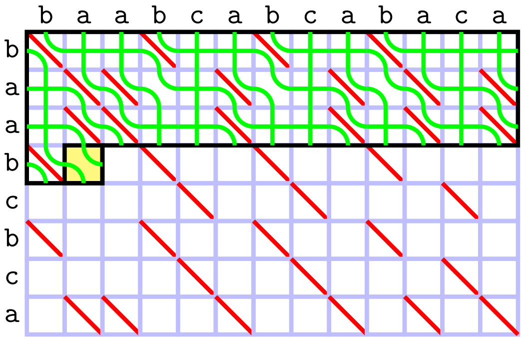
b a a b c a b c a b a c a
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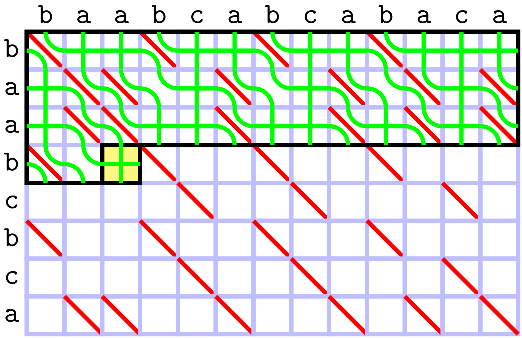
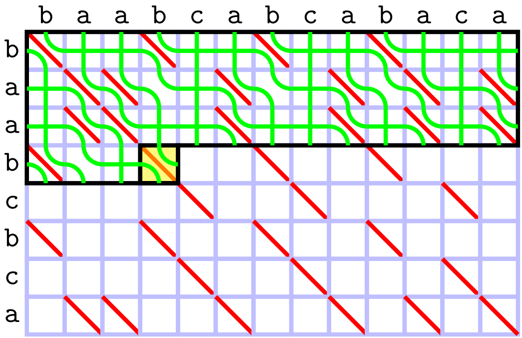
The seaweed algorithm

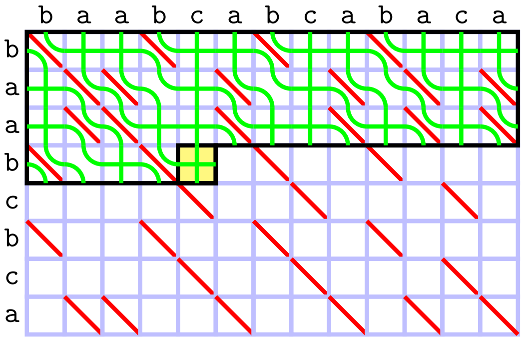
The seaweed algorithm



The seaweed algorithm

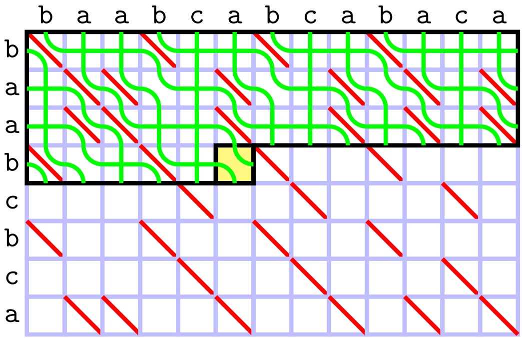
The seaweed algorithm




b a a b c a b c a b a c a
b a a b c b c a

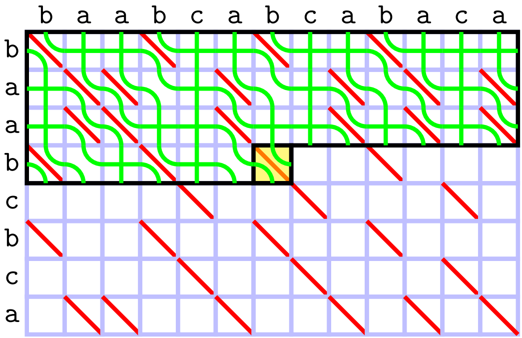
The seaweed algorithm

The seaweed algorithm



The seaweed algorithm

The seaweed algorithm



The seaweed algorithm

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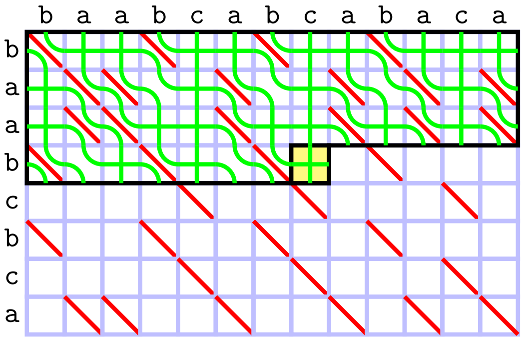
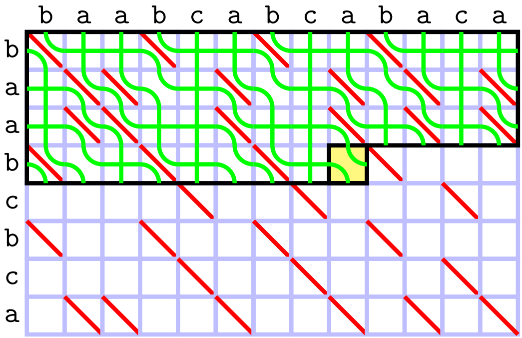


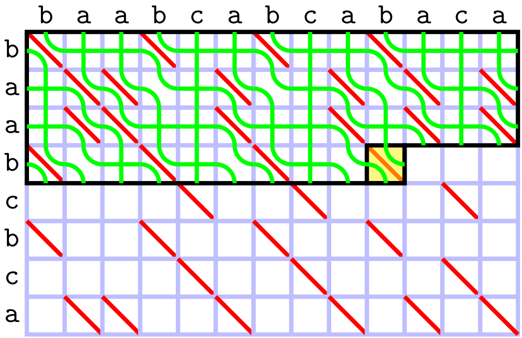
The seaweed algorithm

The seaweed algorithm



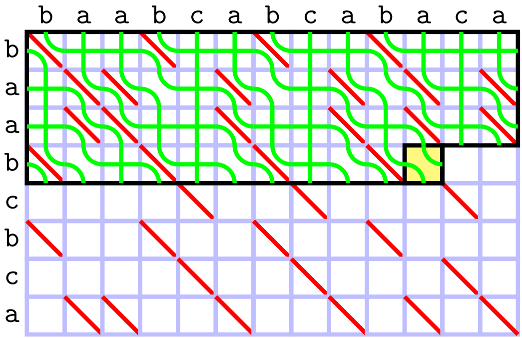
The seaweed algorithm

The seaweed algorithm



The seaweed algorithm

The seaweed algorithm



The seaweed algorithm

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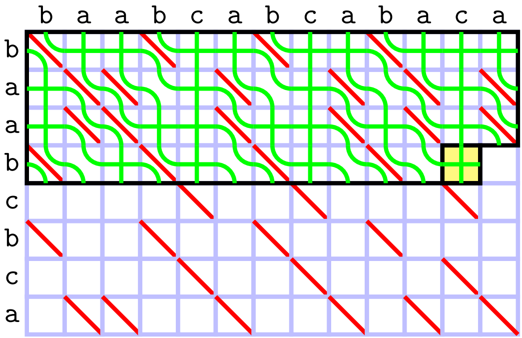

The seaweed algorithm

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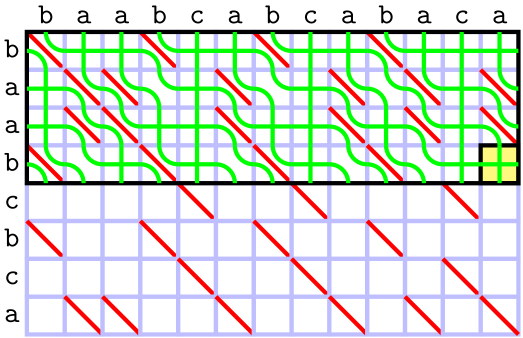
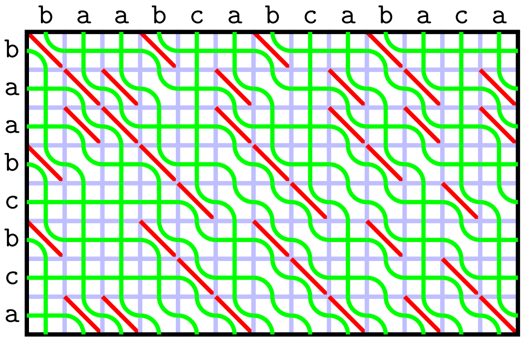


The seaweed algorithm

The seaweed algorithm



Semi-local LCS: the seaweed algorithm [T: 2006]

Iterate over alignment graph, tracing seaweeds

Pick cells in any order, respecting dependencies

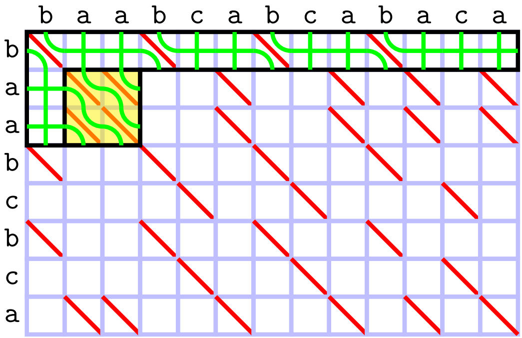
In every cell, the two entering seaweeds

- cross, if mismatch and they have not crossed before
- bend otherwise

Running time $O(mn)$

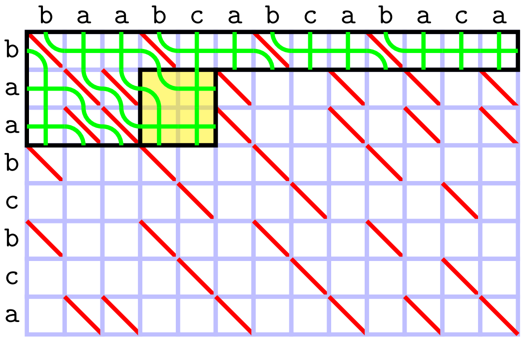
The seaweed algorithm

The micro-block seaweed algorithm



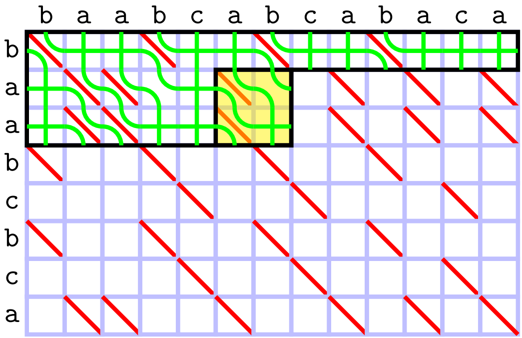
The seaweed algorithm

The micro-block seaweed algorithm



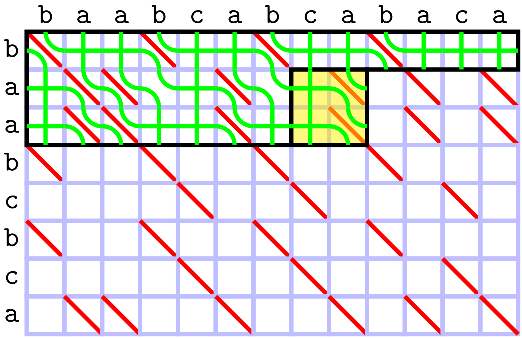
The seaweed algorithm

The micro-block seaweed algorithm



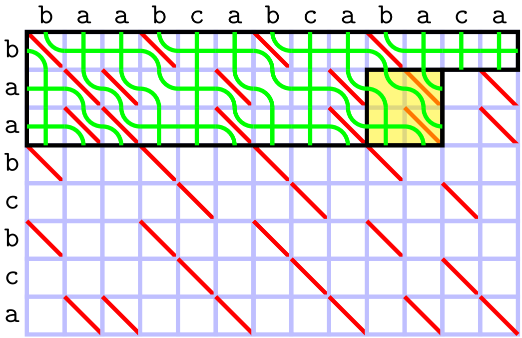
The seaweed algorithm

The micro-block seaweed algorithm



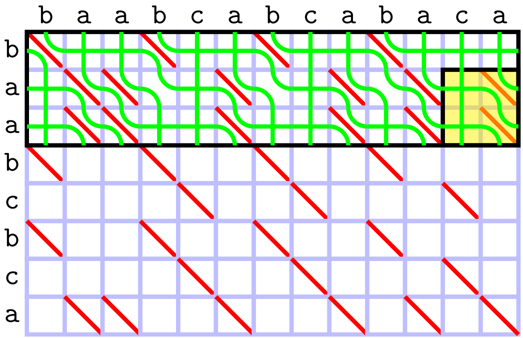
The seaweed algorithm

The micro-block seaweed algorithm



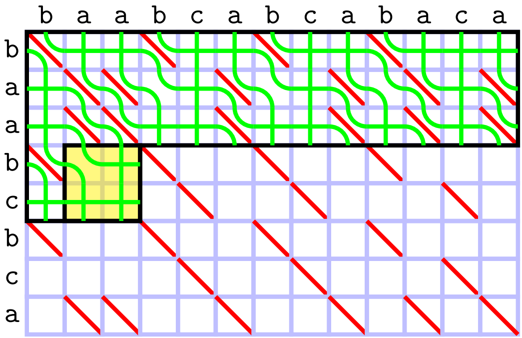
The seaweed algorithm

The micro-block seaweed algorithm



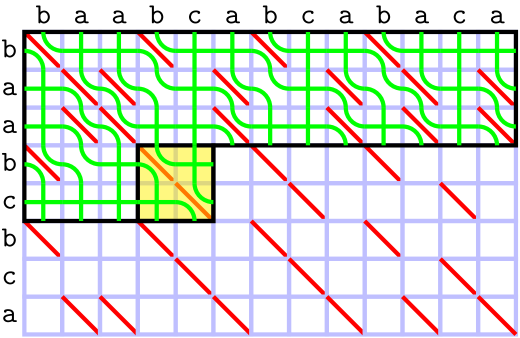
The seaweed algorithm

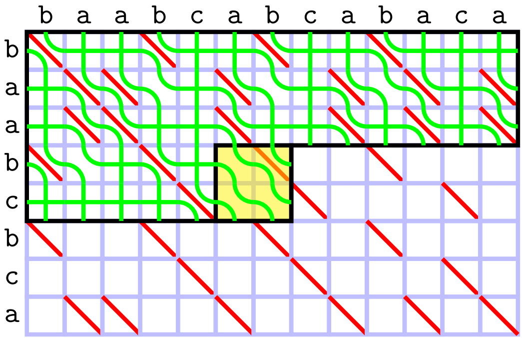
The micro-block seaweed algorithm



The seaweed algorithm

The micro-block seaweed algorithm




b a a b c a b c a b a c a
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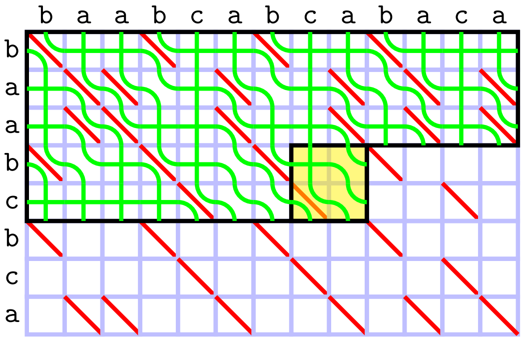
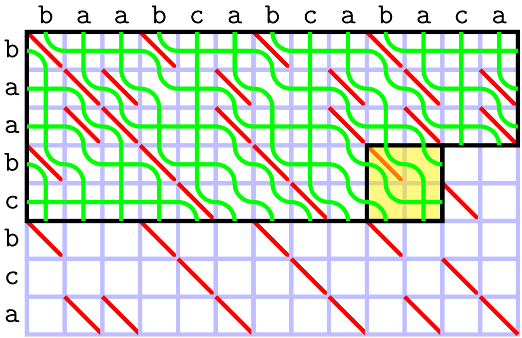
The seaweed algorithm

The micro-block seaweed algorithm




b a a b c a b c a b a c a
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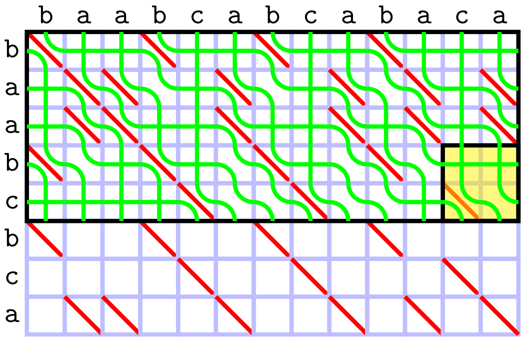
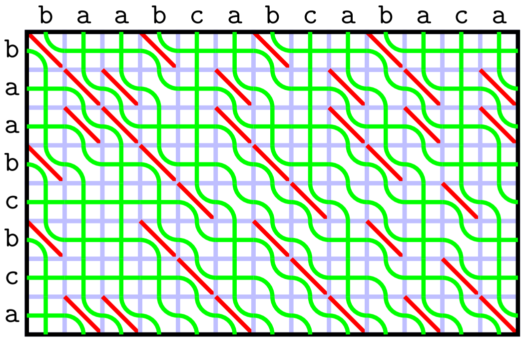
The seaweed algorithm

The micro-block seaweed algorithm



The seaweed algorithm

The micro-block seaweed algorithm



Semi-local LCS: the micro-block seaweed algorithm [T: 2007]

Iterate over alignment graph in blocks, tracing seaweeds

Use precomputed mapping of all possible block inputs to outputs

Block size: $t = O\left(\frac{\log n}{\log\log n}\right)$

Block interface: $O(t)$ values (input chars and seaweeds), each of size $O(\log n)$ but can be compressed to $O(\log\log n)$ by a recursive scheme

Running time $O\left(\frac{m}{t} \cdot \frac{n}{t} \cdot t \log\log n\right) = O\left(\frac{mn(\log\log n)^2}{\log n}\right)$, even on log-cost RAM

The cyclic LCS problem

Give the maximum LCS score for a vs all cyclic rotations of b

The cyclic LCS problem

Give the maximum LCS score for a vs all cyclic rotations of b

Cyclic LCS: running time

$O\left(\frac{mn^2}{\log n}\right)$	naive
$O(mn \log m)$	[Maes: 1990]
$O(mn)$	[Bunke, Bühler: 1993; Landau+: 1998; Schmidt: 1998]
$O\left(\frac{mn(\log\log n)^2}{\log n}\right)$	[T: 2007]

Cyclic LCS: the algorithm

Run the micro-block seaweed algorithm on a vs bb, time $O\left(\frac{mn(\log\log n)^2}{\log n}\right)$

Make n string-substring LCS queries, time negligible

The longest repeating subsequence problem

Find the longest subsequence of a that is a square (a repetition of two identical strings)

Motivated by "tandem repeats" in genome

The longest repeating subsequence problem

Find the longest subsequence of a that is a square (a repetition of two identical strings)

Motivated by "tandem repeats" in genome

Longest repeating subsequence: running time

$O(n^3)$	naive
$O(n^2)$	[Kosowski: 2004]
$O\left(\frac{n^2(\log\log n)^2}{\log n}\right)$	[T: 2007]

Longest repeating subsequence: the algorithm

Run the micro-block seaweed algorithm on a vs bb, time $O\left(\frac{mn(\log\log n)^2}{\log n}\right)$

Make $n-1$ suffix-prefix LCS queries, time negligible

The approximate pattern matching problem

Give the substring closest to a by alignment score, starting at each position in b

Assume rational alignment score

Approximate pattern matching: running time

$O(mn)$		[Sellers: 1980]
$O\left(\frac{mn}{\log n}\right)$	$\sigma = O(1)$	via [Masek, Paterson: 1980]
$O\left(\frac{mn(\log\log n)^2}{\log n}\right)$		via [Paterson, Dancik: 1994]

Approximate pattern matching: the algorithm

Run the micro-block seaweed algorithm on a vs b under given alignment score in time $O\left(\frac{mn(\log\log n)^2}{\log n}\right)$

The implicit semi-local edit score matrix:

- an anti-Monge matrix
- approximate pattern matching $\sim$ row minima

Row minima in $O(n)$ element queries [Aggarwal+: 1987]

Each query in time $O(\log^2 n)$ using the range tree representation, combined query time negligible

Overall running time dominated by block seaweed algorithm, same as [Paterson, Dancik: 1994]

The periodic string-substring LCS problem

Give (implicit) LCS scores for a vs each substring of $b = \ldots uuu \ldots = u^{\pm\infty}$

Let u be of length p; may assume that every character of a occurs in u

The tandem LCS problem

Give LCS score for a vs $b = u^k$

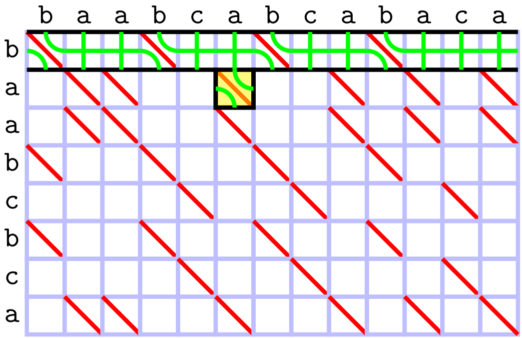
We have $n = kp$; may assume $k \leq m$ (otherwise LCS score is m)

Tandem LCS: running time

$O(mkp)$	naive
$O(m(k+p))$	[Landau, Ziv-Ukelson: 2001]
$O(mp)$	[NEW]

The seaweed algorithm

The periodic seaweed algorithm



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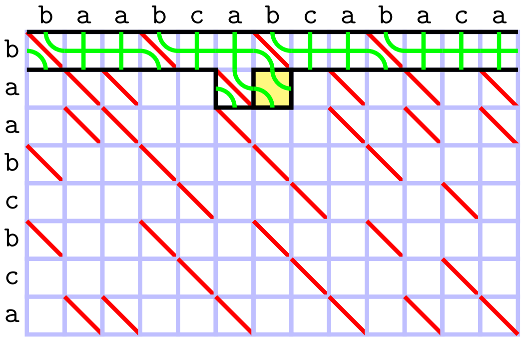

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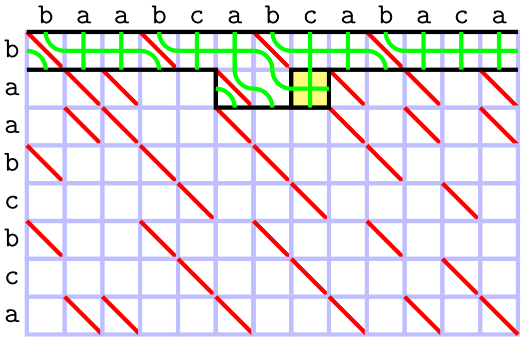

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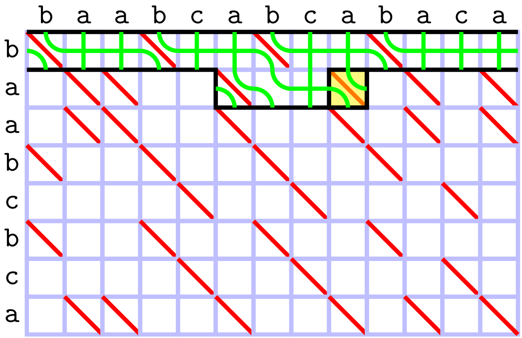

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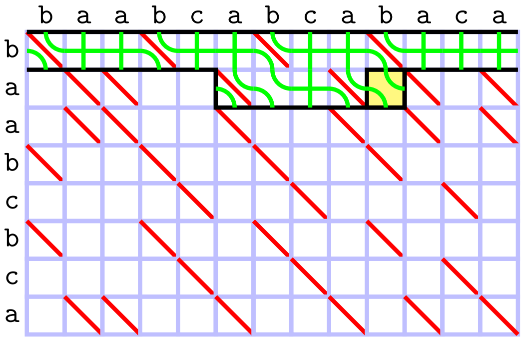

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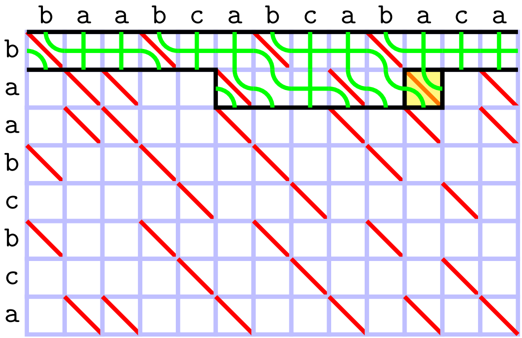

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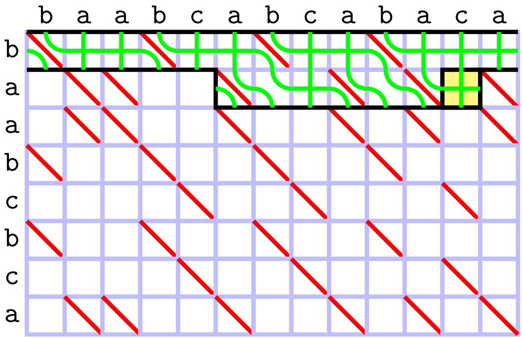

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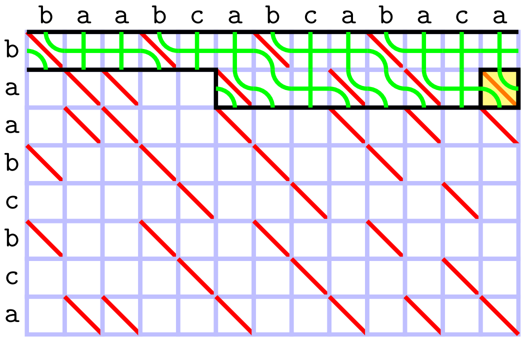

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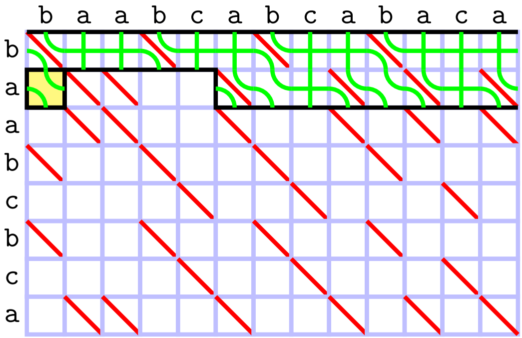

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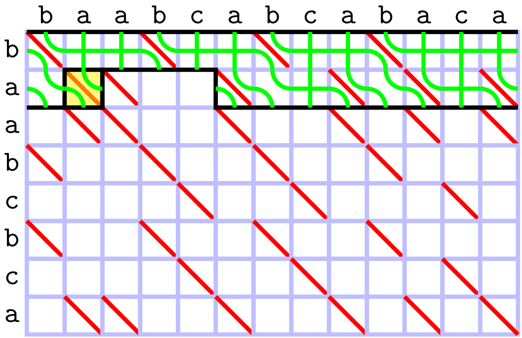

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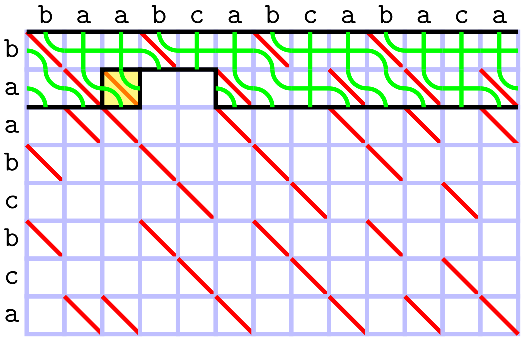

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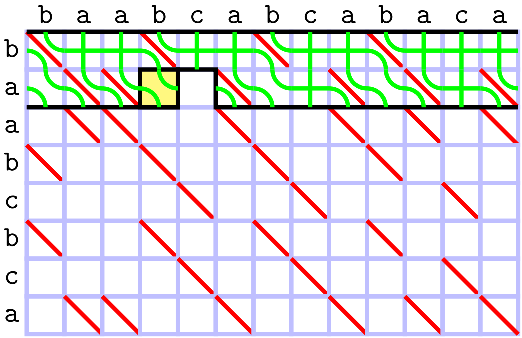

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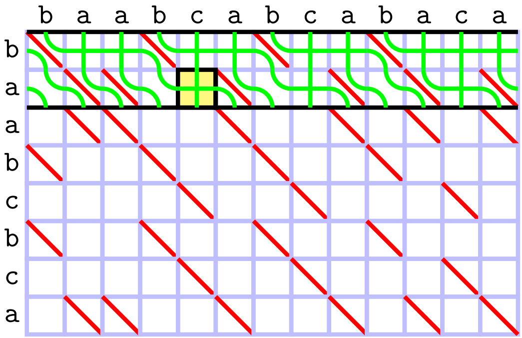

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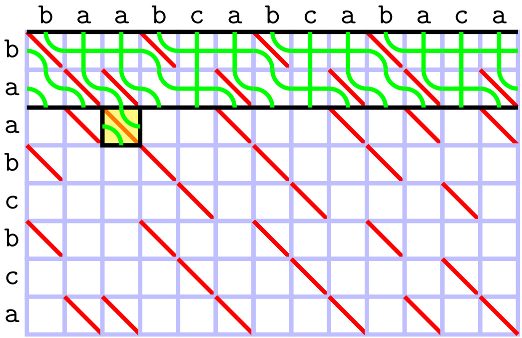

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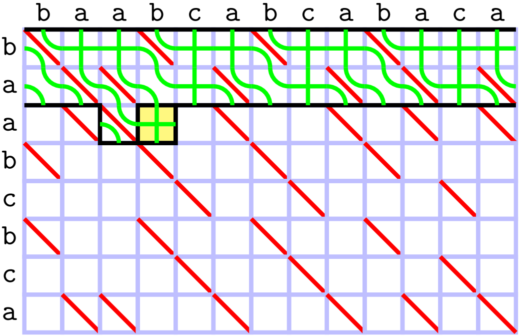

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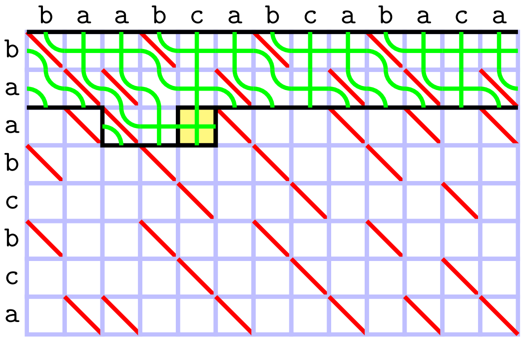

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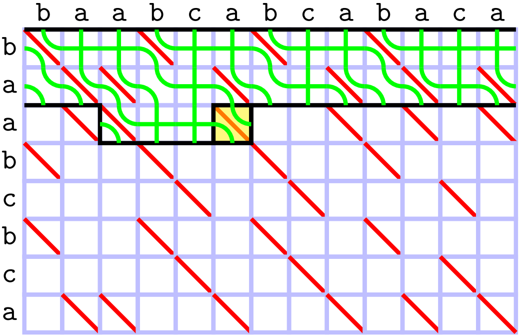

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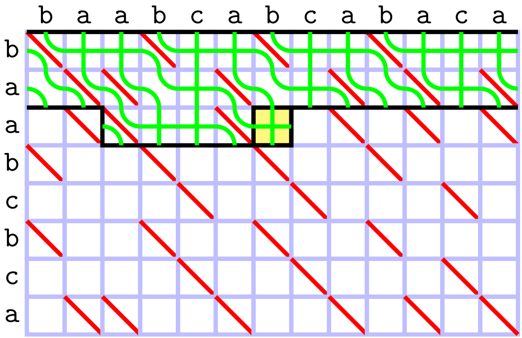

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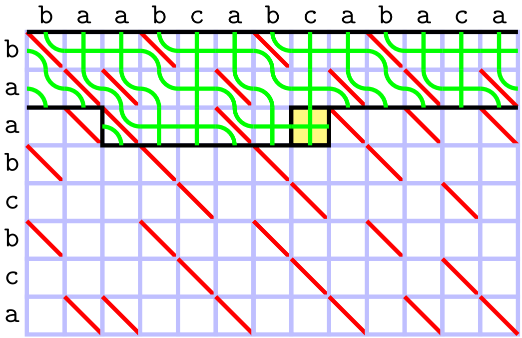

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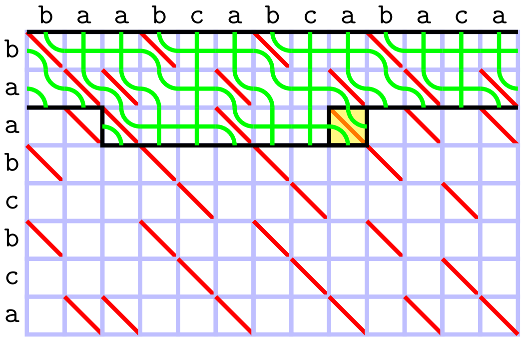

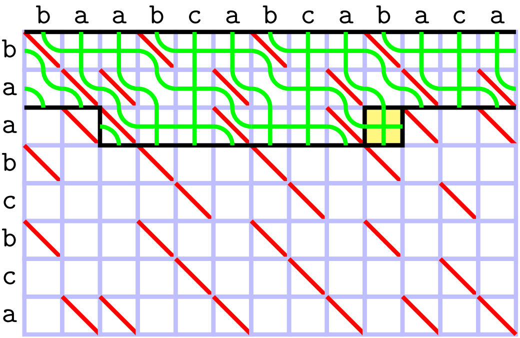
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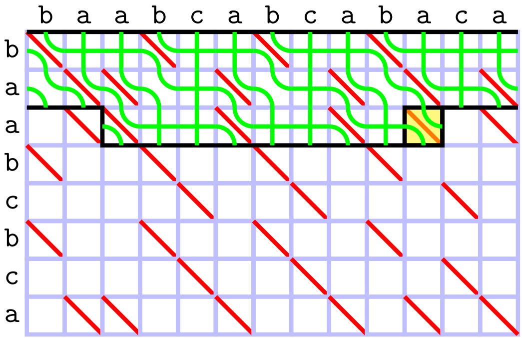

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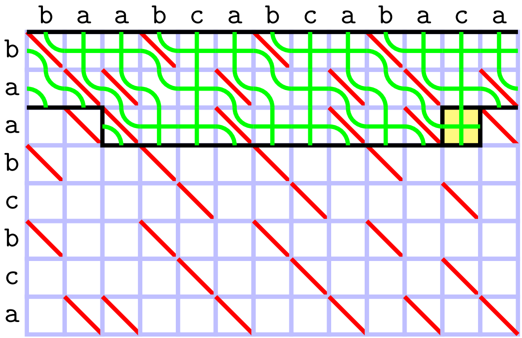

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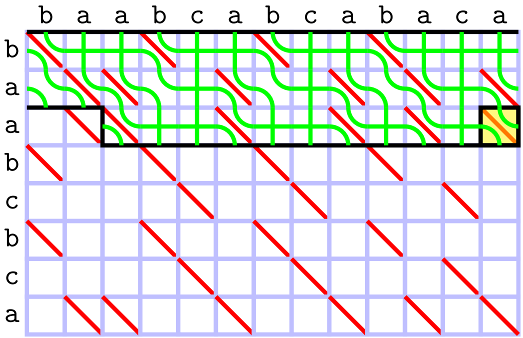

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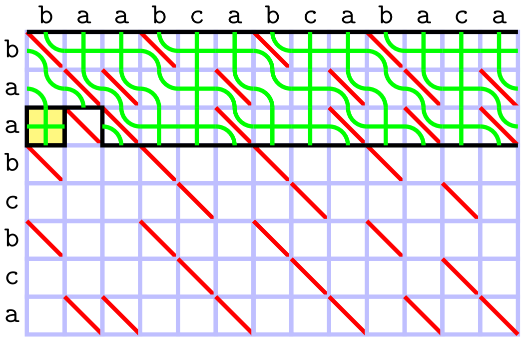

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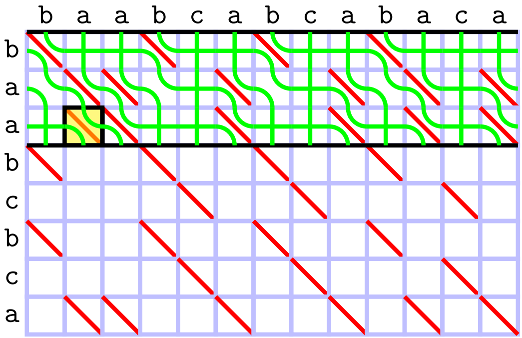

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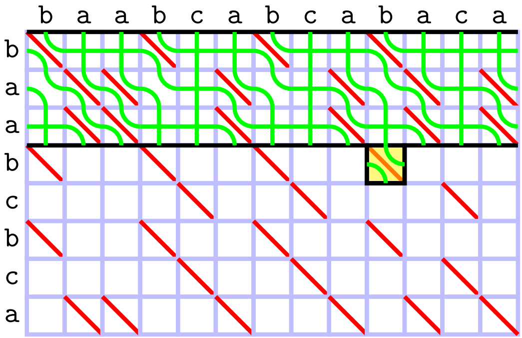

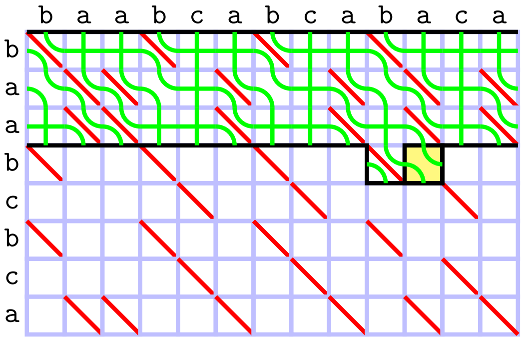
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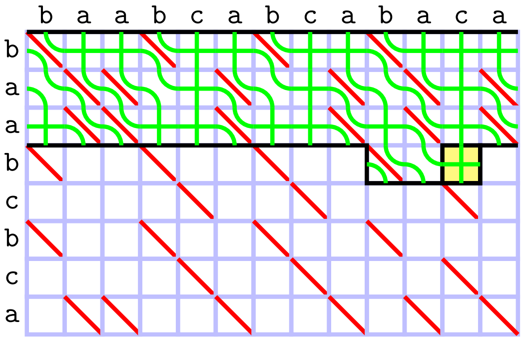

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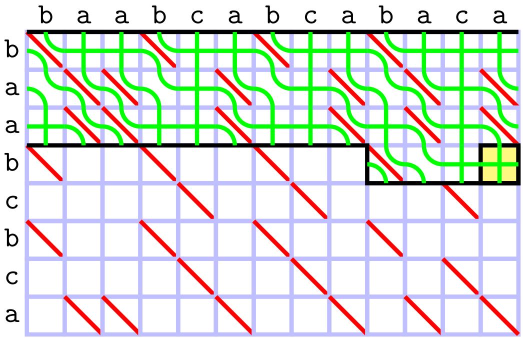

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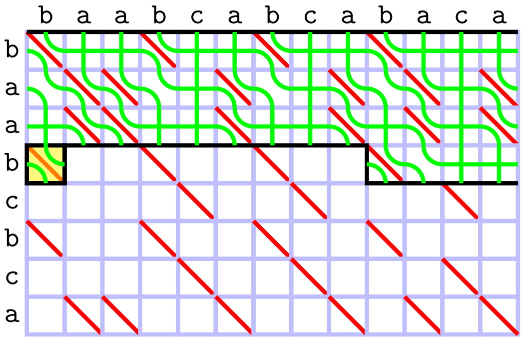

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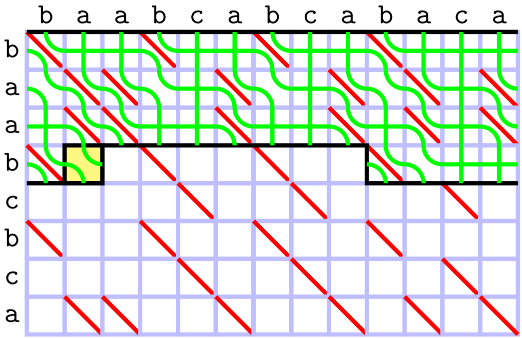

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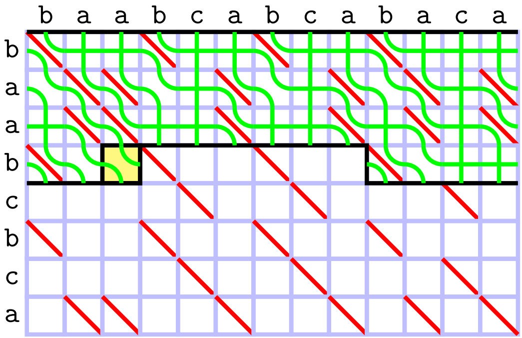

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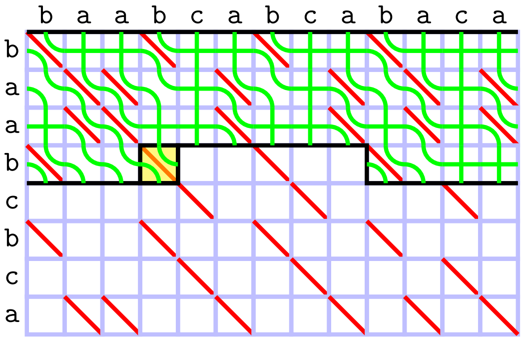

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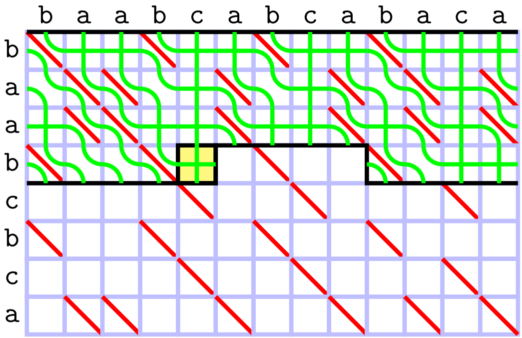

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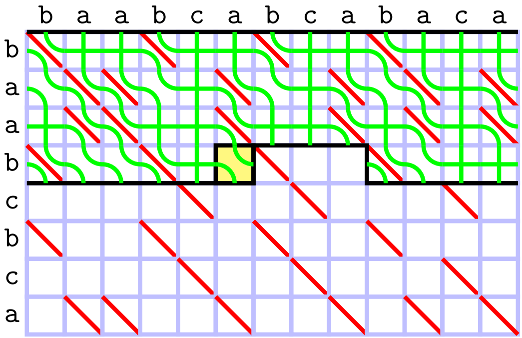

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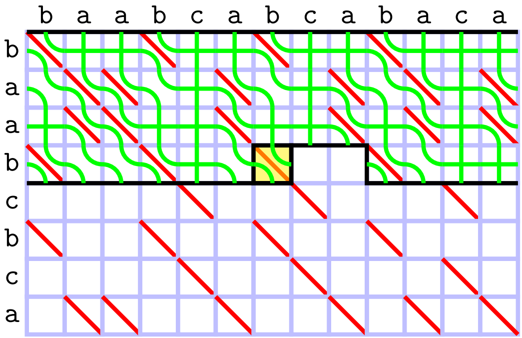

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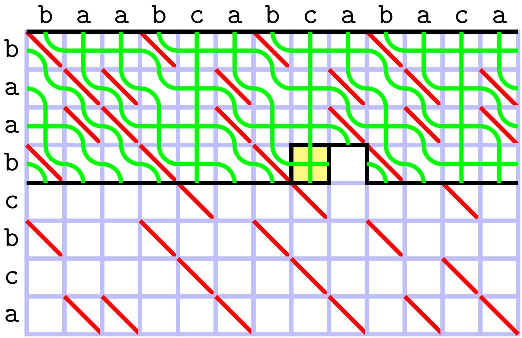

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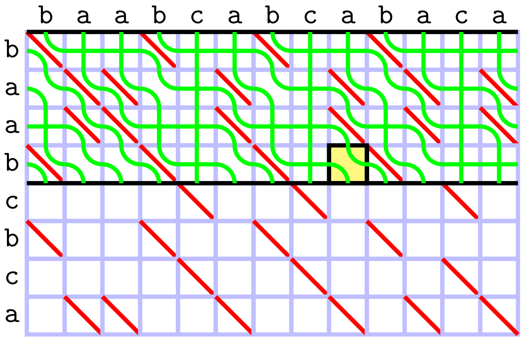

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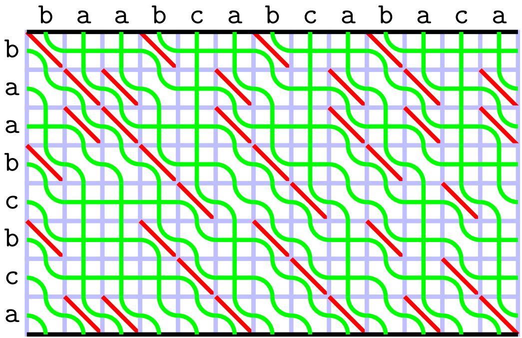

Periodic string-substring LCS: The periodic seaweed algorithm

Iterate over alignment graph, tracing seaweeds row-by-row

In every row, start from a match cell and move rightwards, wrapping around at the graph edge

In every cell, the two entering seaweeds

- cross, if mismatch and they have not crossed before
- bend otherwise
- at right edge of the graph, wrap around back to left edge

Running time $O(mn)$

Querying a string-substring LCS score (including tandem LCS): count each nonzero dominated by query point with appropriate multiplicity, either directly or via a range tree

The tandem alignment problem

Give the substring closest to a by alignment score among certain substrings of $b = u^{\pm\infty}$:

- global: substrings of the form b^k across all k
- cyclic: substrings of length kp across all k
- local: substrings of any length

Tandem alignment: running time

$O(m^2 p)$	all	naive
$O(mp)$	global	[Myers, Miller: 1989]
$O(mp \log p)$	cyclic	[Benson: 2005]
$O(mp)$	cyclic	[NEW]
$O(mp)$	local	[Myers, Miller: 1989]

Cyclic tandem alignment: the algorithm

Run periodic seaweed algorithm (under given alignment score), time $O(np)$

For each $k \in [1 : m]$:

- solve tandem LCS (under given alignment score) for a against b^k
- obtain p successive string-substring alignment scores by inremental score updating, each in time $O(1)$

Running time $O(mp)$

Conclusions and future work

Semi-local LCS problem:

- representation by implicit unit-Monge matrices
- generalisation to rational alignment scores
- open: real alignment scores?

The seaweed and micro-block seaweed algorithms:

- a simple algorithm for semi-local LCS
- semi-local LCS in time $o(mn)$ via micro-blocks
- improvements on related problems

The periodic seaweed algorithm:

- a straightforward extension of the seaweed algorithm
- periodic semi-local LCS in time $O(mp)$
- natural applications
- open: $o(mp)$ via micro-blocks?