Tiling an interval of the discrete line

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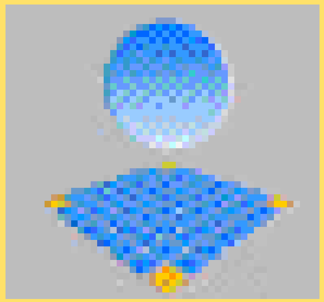

Discrete Tiling

- space :

- pattern :

- find the translation set or dual:
  the positions in the space where to put copies of the pattern to cover the whole space

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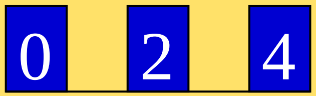

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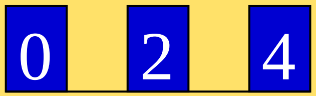

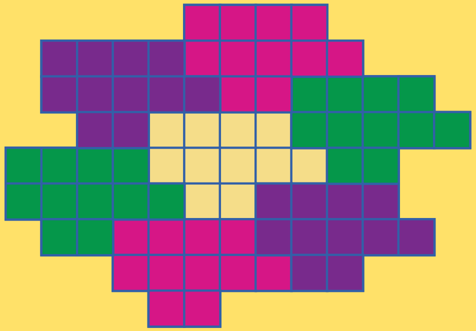

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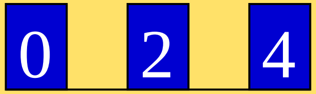

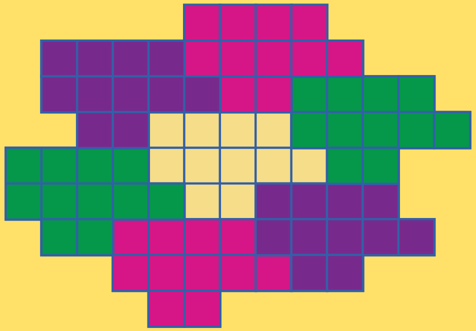

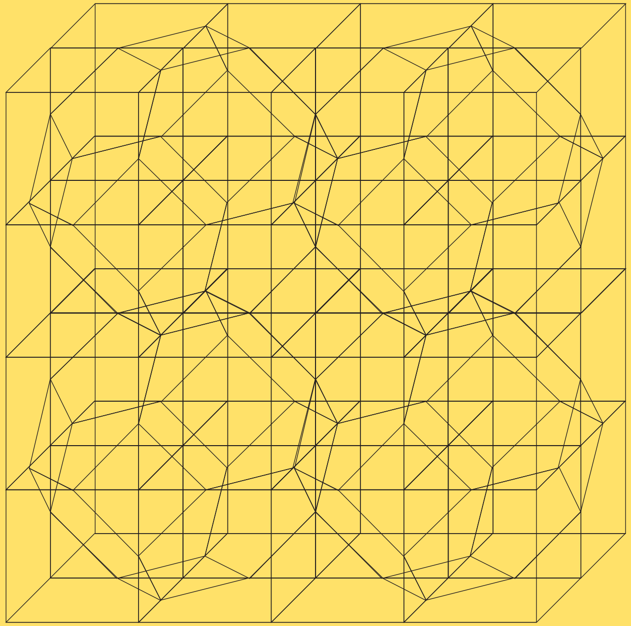

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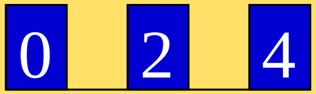

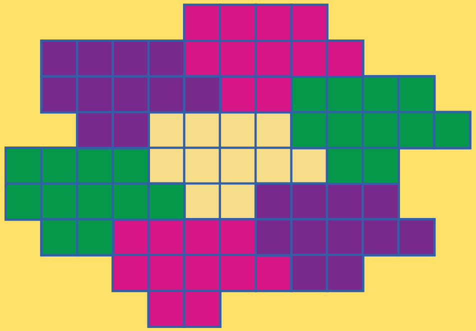

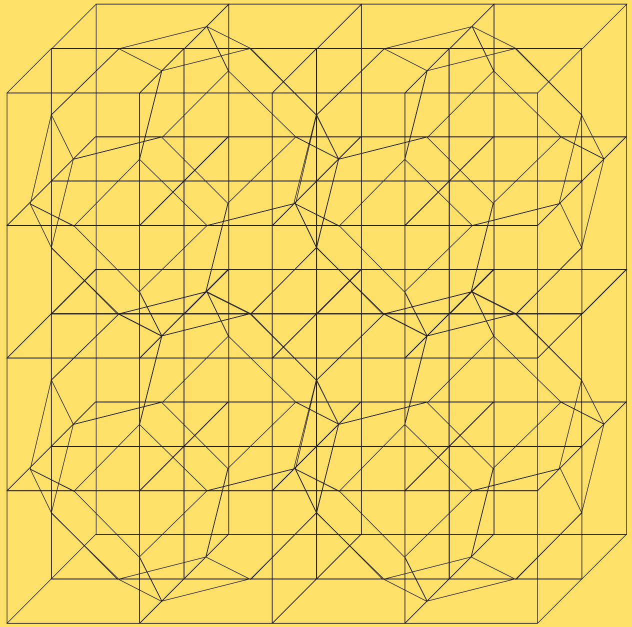

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➢ in our case, the space is the discrete line: $\mathbb{Z}$ (*i.e.*, $\{-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, +\infty\}$)

Tiling the 2D space with polyominos [*Culik 1996*]

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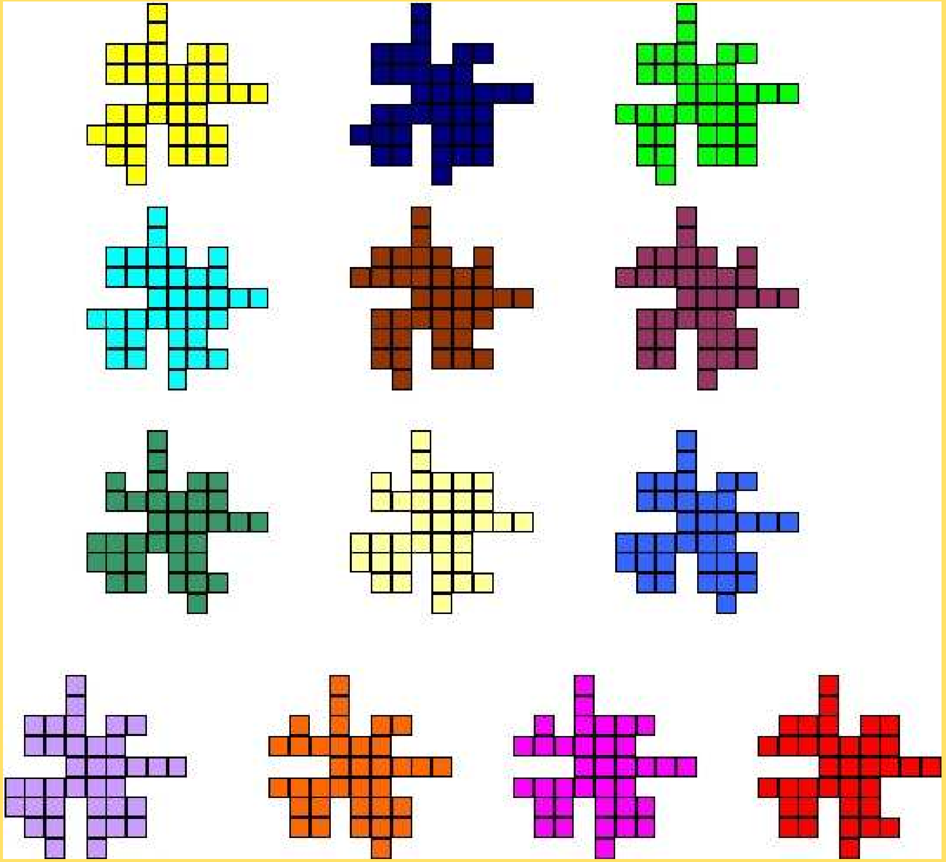

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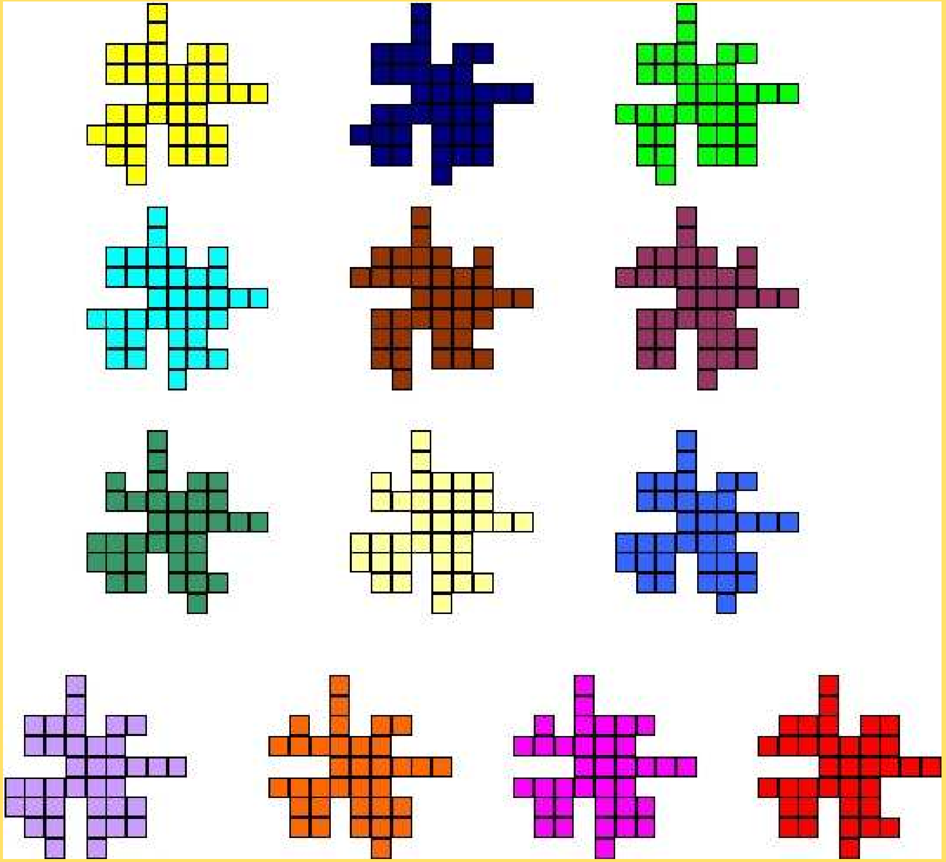

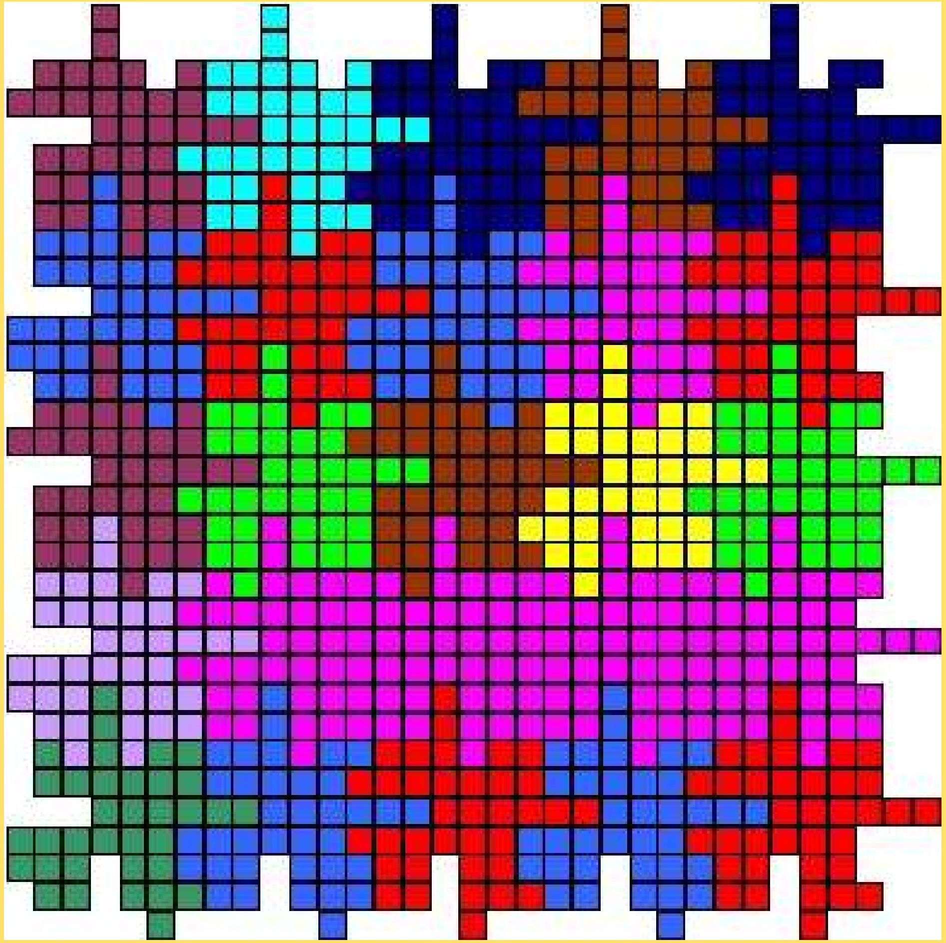

Periodicity of tilings of $\mathbb{Z}$ [*Lagarias and Wang, 1996*]

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- One can check in exponential time whether a pattern A tiles $\mathbb{Z}$.
- However, in all examples, $k \leq 2d(A)$ [*Nivat's conjecture*].

Notations

Let A, B be subsets of $\mathbb{Z}$ and let $k \in \mathbb{N}$:

- Hyp: 0 belongs to A, and $A \subseteq \mathbb{N}$

  always true if A can be translated

- $\#(A)$ the *cardinality* of A,

- $A + k := \{a + k : a \in A\}$ is a set, a *translate* of A

- $[\![k]\!]$ the *interval* $[0, k-1]$

- $A \uplus B := \cup_{b \in B} A + b$ is a *multi-set*
  Example: $\{0,1,4\} \uplus \{0,2,5\} = \{0,1,2,3,4,5,6,6,9\}$; 6 occurs twice.

- if it is a set, then $A \uplus B$ is denoted $A \oplus B$, the direct sum of A and B
  Example: $\{0,1,4\} \uplus \{0,2\} = \{0,1,2,3,4,6\}$

Tiling : definition

Let $n \geq 0$. Let X and f be subsets of $\mathbb{Z}$.

Tiling, dual : f *tiles* X if and only if there exists $\hat{f}_X$, a subset of $\mathbb{N}$, such that $f \oplus \hat{f}_X = X$.

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Two trivial tiles: $f := [0, n-1] = [\![n]\!]$ or $f := \{0\}$

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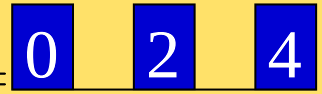
$f =$ 

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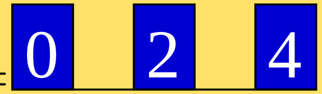
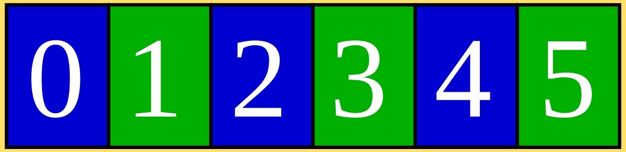
$f =$ 



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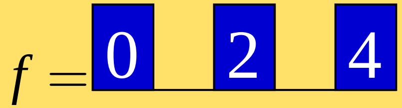

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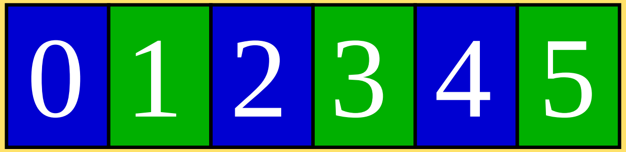

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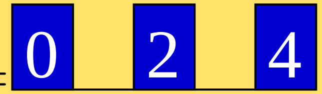
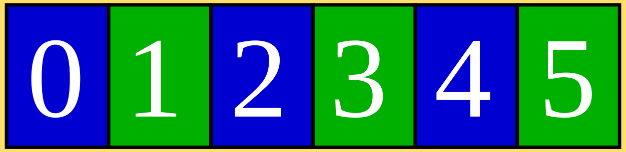
  $f =$ 

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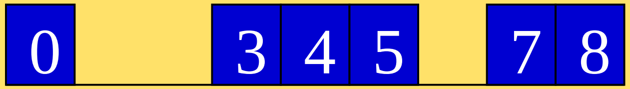
  $f =$ 

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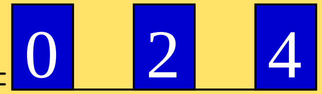
$f =$ 

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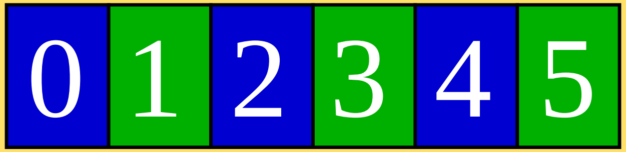

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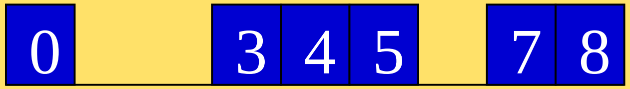
$f =$ 

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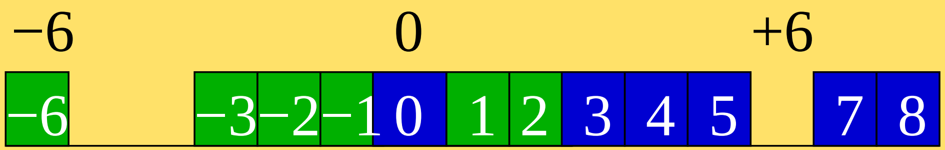

f also tiles $\mathbb{Z}/12\mathbb{Z}$.

Some properties of the tiles of $[\![n]\!]$.

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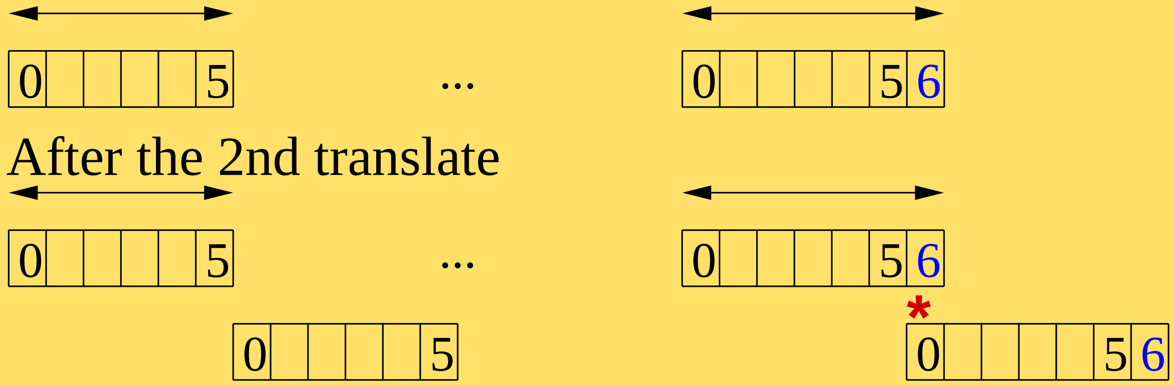

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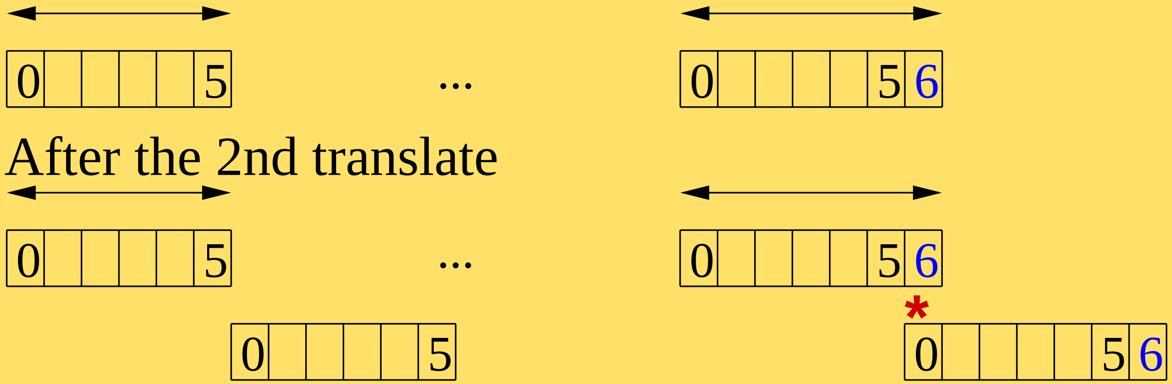

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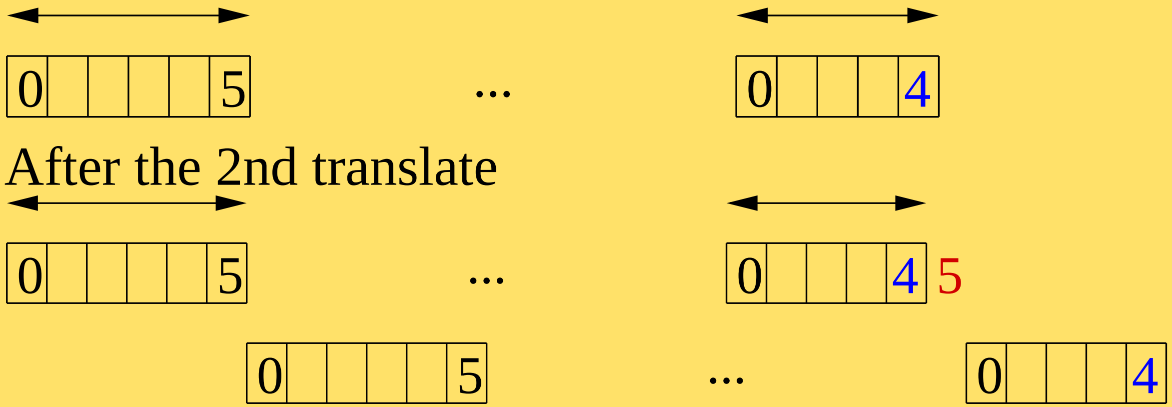

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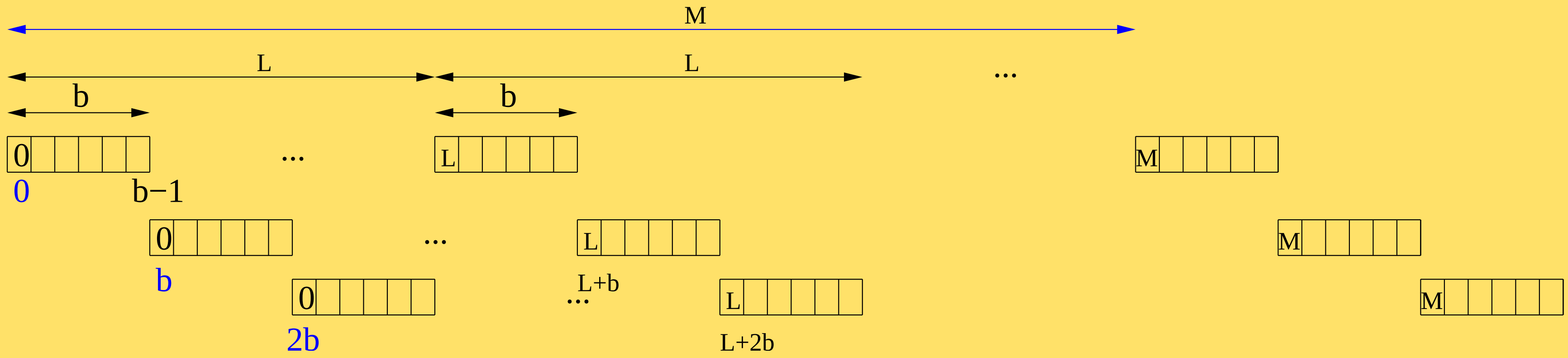

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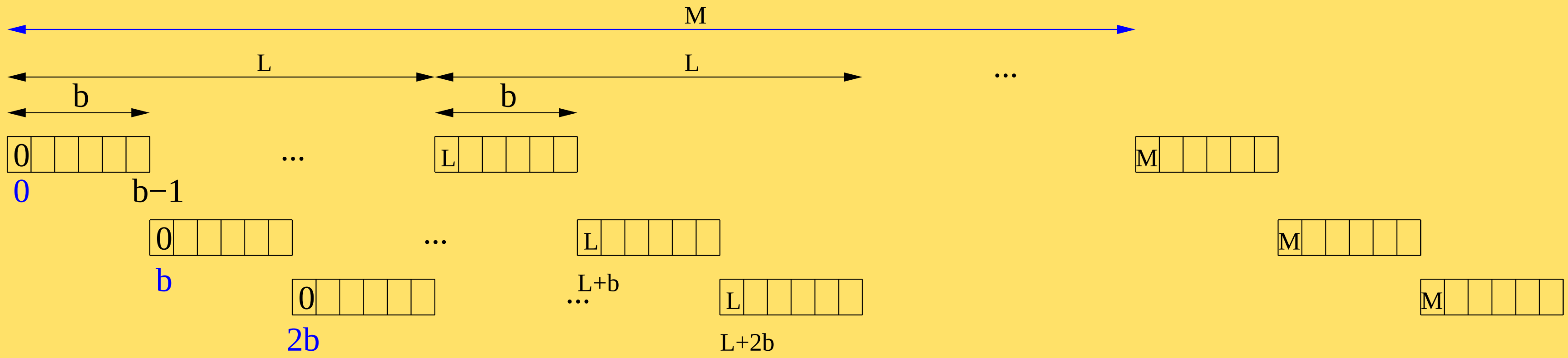

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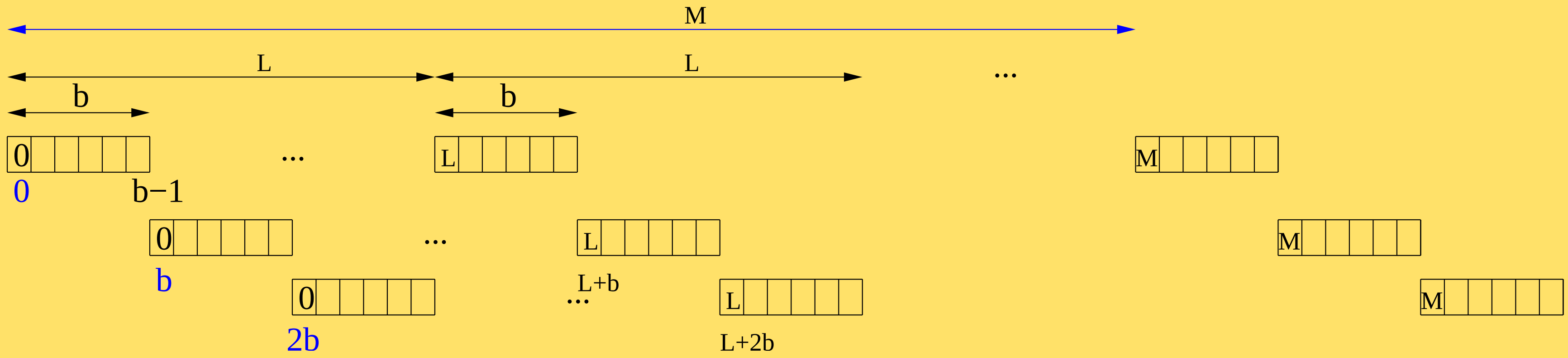

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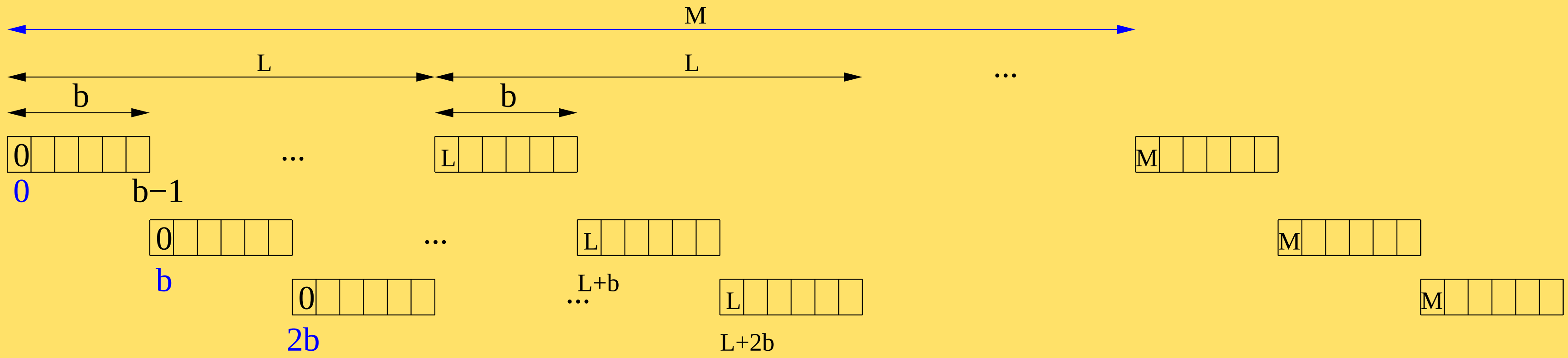

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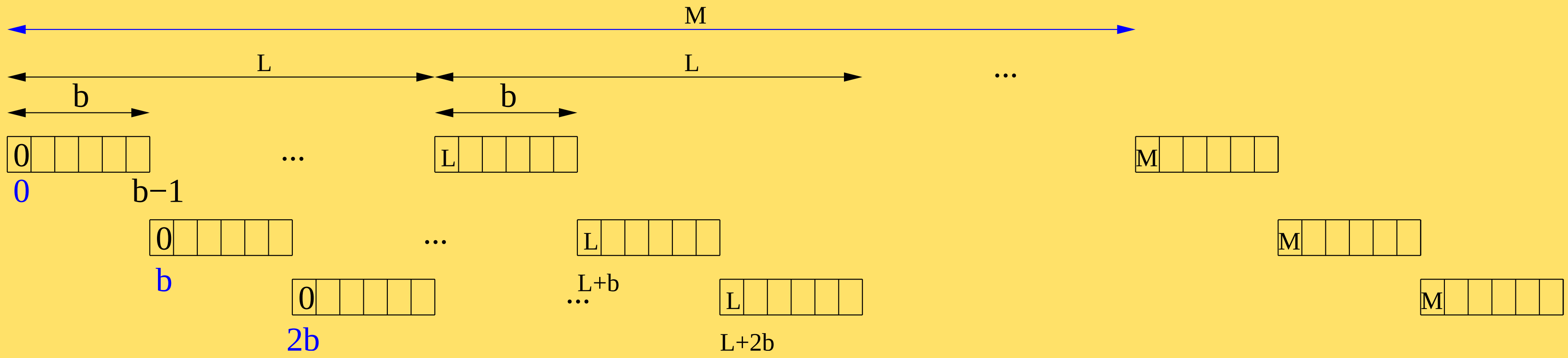

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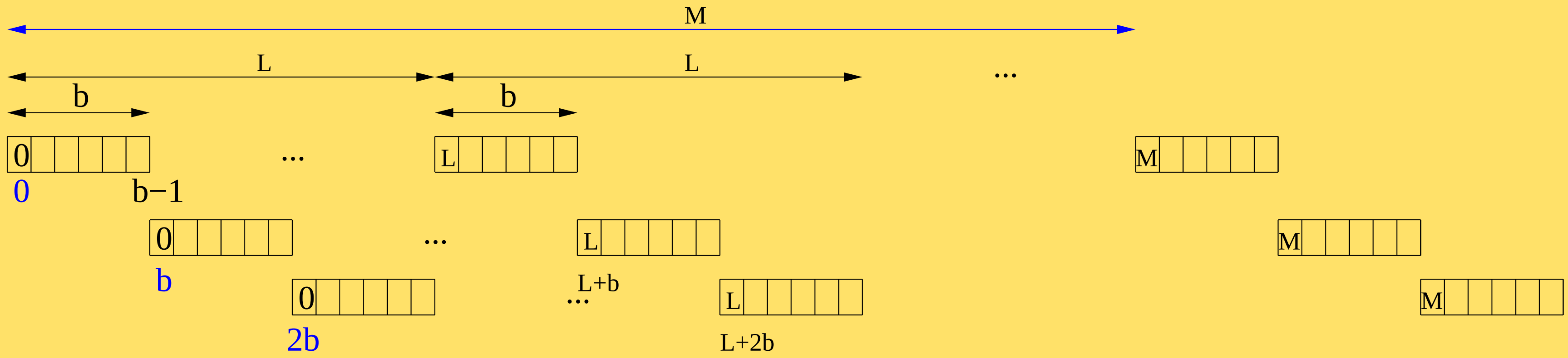

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- The sequence of new block offsets is strictly increasing

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- *Example*:

$$\begin{aligned} f &= [\![7]\!] \oplus \{0,21,42,126,147,168,504,525,546,630,651,672\} \\ &= [\![7]\!] \oplus \{0,21,42\} \oplus \{0,126,504,630\} \\ &= [\![7]\!] \oplus \{0,21,42\} \oplus \{0,126\} \oplus \{0,504\}\ . \end{aligned}$$

Self-period and tiles

Self periodicity of a pattern

Let $n \geq 0$ and f be a pattern such that $d(f) < n$.

Self-period of a pattern : Let p be an integer such that $0 \leq p \leq d(f)$.
p is a *self-period of f for length n* if and only if for any $i \in [0, n-p[$ one has

$$i \in f \Leftrightarrow (i+p) \in f \ .$$

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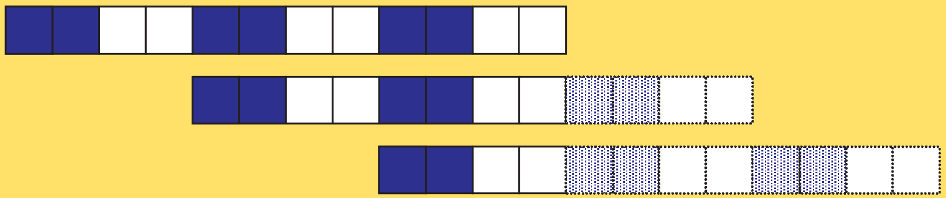


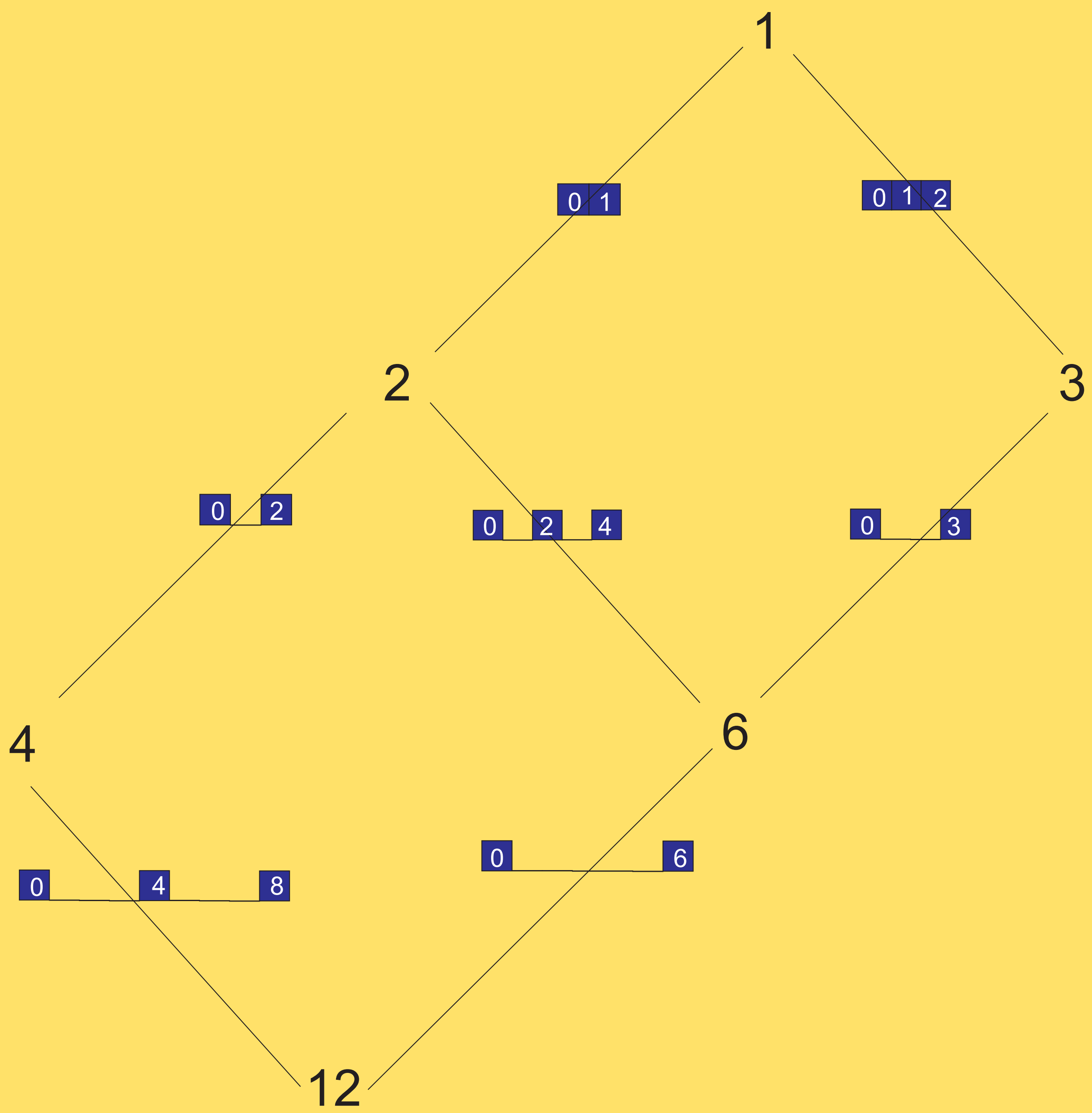
Example

- Let $n := 12$ and $f := \{0,1,4,5,8,9\}$.
- f tiles $[\![12]\!]$; its dual for $n := 12$ is $\hat{f}_{12} := \{0,2\}$
- $\#(f) \times \#\left(\hat{f}_{12}\right) = 6 \times 2 = 12$.
- $\hat{f}_{12}$ tiles $[\![4]\!]$, $[\![8]\!]$, and $[\![12]\!]$.
- f has periods 0, 4, and 8. So, $\pi_{12}(f) = 4$ and $\Pi_{12}(f) = \{0,4,8\}$.
- f can be decomposed in $\{0,1,4,5,8,9\} = \{0,1\} \oplus \{0,4,8\}$.
- $\{0,1\}$ and $\{0,4,8\}$ are completely periodic for lengths 2 and 12 resp., with smallest period 1 and 4 resp.

Properties of the self-period of a tile

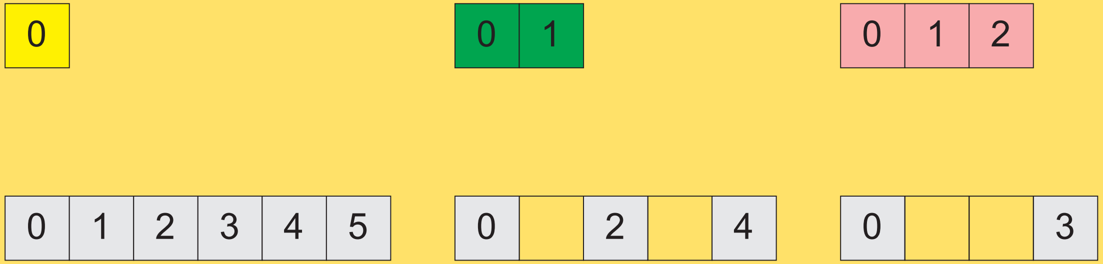
- f admits a smallest non null period $\pi_n(f)$
- $\pi_n(f)$ divides n and thus, $\pi_n(f) \leq \lfloor n/2 \rfloor$
- $f[\pi_n(f)] \oplus \hat{f}_n = [\![\pi_n(f)]\!]$
- *Theorem*: there exists a bijection that to any pattern f such that $d(f) \leq n/2$, associates a pattern that tiles $[\![d]\!]$ for d a divisor of n.

Decomposition of a tile in completely periodic tiles



Bijection

For $n := 6$



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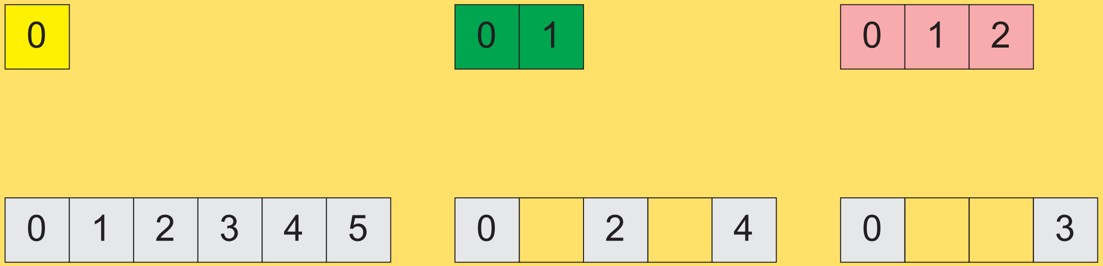
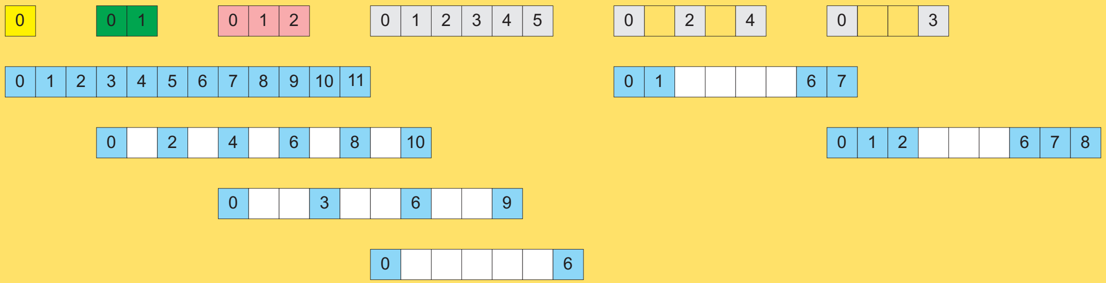


For $n := 12$



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- Sequence A067824 of the On Line Encyclopedia of Integer Sequence Sequence:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
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- Proof that sequence $A067824$ is identical to sequence $A107736$

Conclusion

Results:

- Optimal recognition algorithm
- Bound on the periodicity
- Counting formula
- Polynomial representation

Future work:

- generalize to higher dimensions
- with more than one tile
- the torus $\mathbb{Z}/n\mathbb{Z}$ [*Minkowski problem*]
  Example: $f = \{0, 8, 16, 18, 26, 34\}$ tiles $\mathbb{Z}/72\mathbb{Z}$
  with dual $\{0, 1, 5, 6, 12, 25, 29, 36, 42, 48, 49, 53\}$

Thanks for your attention