Text Indexing with Errors

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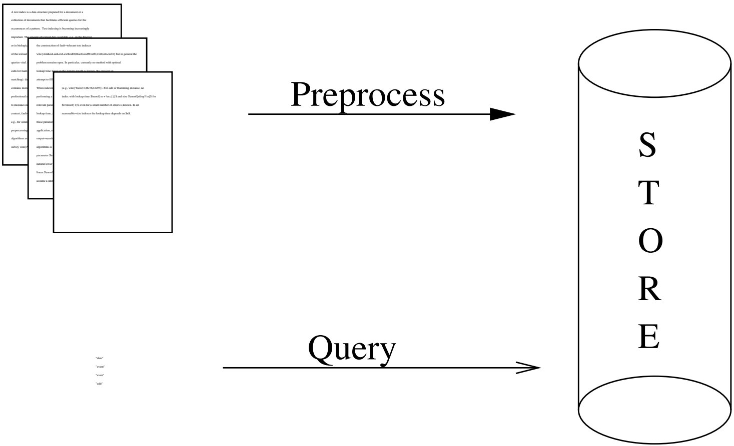
June 20, 2005

Overview



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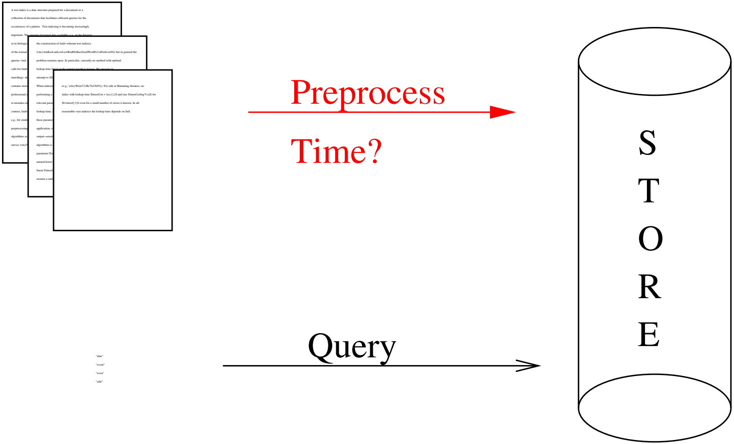

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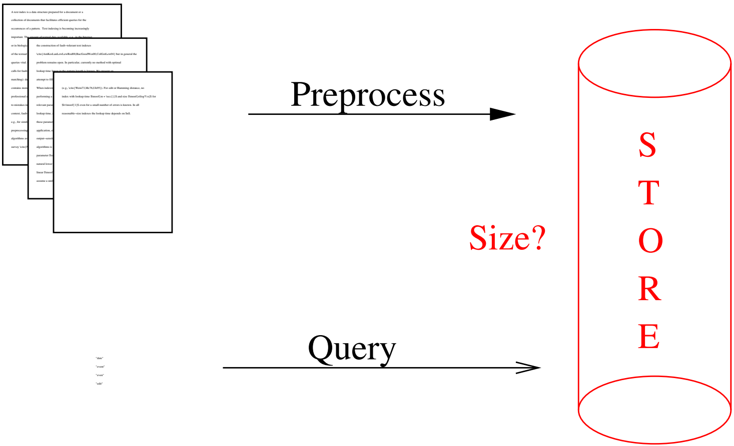

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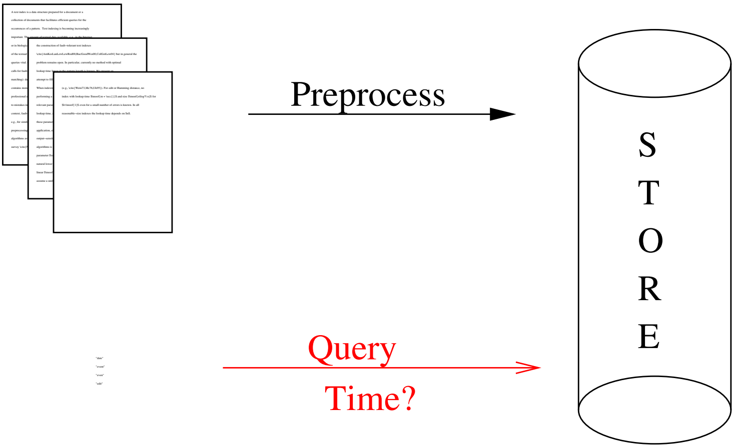

Indexing Problems

One pattern is queried against

- a single document – Text Indexing
- a collection of words
- a collection documents

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Previous Work

Indexing a text of size[1] n and querying a pattern of size m resulting in occ occurrences.

Errors	Model	Query Time	Index Size	Prep. Time	Literature
$k = 0$	exact	$O(m + \text{occ})$	$O(n)$	$O(n)$	Weiner 1973
$k = 1$	edit	$O(m \log n \log\log n + \text{occ})$	$O(n \log^2 n)$	$O(n \log^2 n)$	Amir et. al 2000
$k = 1$	edit	$O(m \log\log n + \text{occ})$	$O(n \log n)$	$O(n \log n)$	Buchsbaum et. al 2000
$k = 1$	Ham.	$O(m + \text{occ})$	$O(n \log n)$ (avg)	$O(n \log n)$ (avg)	N 2004
$k = O(1)$	edit	$O(m + \log^k n + \text{occ})$	$O(n \log^k n)$	$O(n \log^k n)$	Cole et. al 2004
$k = O(1)$	edit	$O(m \log^2 n + m^2 + \text{occ})$ (avg)	$O(n \log n)$ (avg)	$O(n \log^2 n)$ (avg)	Chávez et al. 2002
$k = O(1)$	edit	$O(m \min\{n, m^{k+1}\} + \text{occ})$	$O(\min\{n, m^{k+1}\} + n)$	$O(\min\{n, m^{k+1}\} + n)$	Cobbs 1995 (Ukkonen 1993)
$k = O(1)$	edit	$O(m + \text{occ})$	$O(n \log^k n)$, (avg,whp)	$O(n \log^{k+1} n)$, (avg,whp)	MN 2005
$k = O(1)$	edit	$O(m + \text{occ})$, (avg,whp)	$O(n \log^k n)$	$O(n \log^{k+1} n)$	
$k = O(1)$	Ham.	$O(\log^{k+1} n)$, (avg)	$O(n)$	$O(n)$	M 2004
$k = \alpha \log n$	Ham.	$O(n^\lambda)$, $\lambda < 1$, (avg)	$O(n)$	$O(n)$	
$k = \alpha m$	edit	$O(n^\lambda \log n)$, $\lambda < 1$	$O(n)$	$O(n)$	Navarro et. al 2000
k mismatches in a window of length r		$O(m + \text{occ})$ (avg)	$O(n \log^l n)$ (avg)	$O(n \log^l n)$ (avg)	Gabriele et. al 2003

[1] Uniform Cost Model

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$k=O(1)$	edit	$O(m \min\{\mathbf{n}, m^{\mathbf{k}+1}\} + \text{occ})$	$O(\min\{\mathbf{n}, m^{k+1}\} + n)$	$O(\min\{n, m^{k+1}\} + n)$	Cobbs 1995 (Ukkonen 1993)
$k=O(1)$	edit	$O(\mathbf{m} + \text{occ})$	$O(n \log^k n)$, (avg,whp)	$O(n \log^{k+1} n)$, (avg,whp)	MN 2005
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Worst-Case Optimal Search-Time

We focus on

- indexing a text t of size n,
- querying a pattern p of size m,
- allowing a constant number of k errors (Edit distance),
- achieving worst-case optimal query-time $O(m + \mathrm{occ})$ and size $O(n \log^k n)$ on average and w.h.p.

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Definition (k-Minimal Prefix Length)

For two strings $u, v \in \Sigma^*$ with $\mathrm{d}(u, v) = k$ we define

$$\mathrm{minpref}_{k,u}(v) = \min \left\{ l \;\middle|\; \begin{array}{l} \mathrm{d}(\mathrm{pref}_l(u), \mathrm{pref}_{l+|v|-|u|}(v)) = k \quad \text{and} \\ \mathrm{suff}_{l+1}(u) = \mathrm{suff}_{l+|v|-|u|+1}(v) \end{array} \right\}.$$

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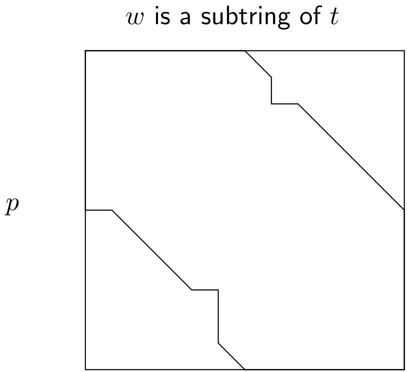
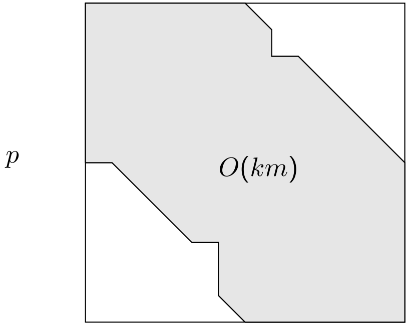
Simple Methods for Indexing



Simple Methods for Indexing

w is a subtring of t



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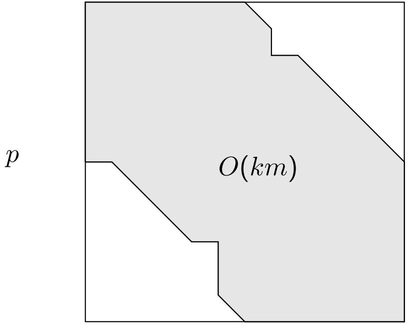
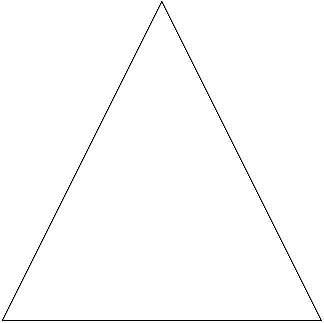


Trie for all strings w within distance k of t



Simple Methods for Indexing

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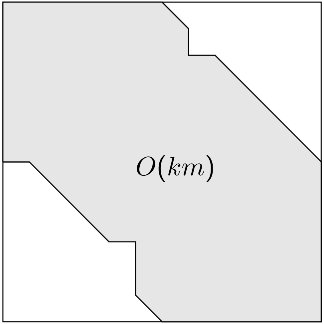
p



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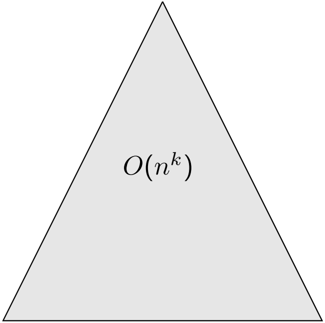
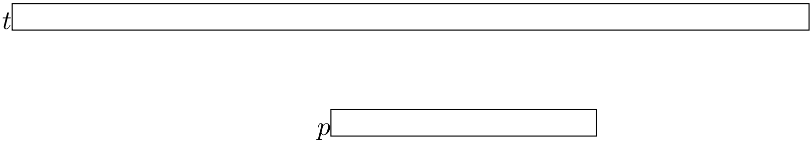


Searching with One Error



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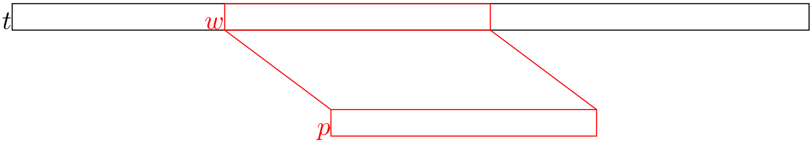

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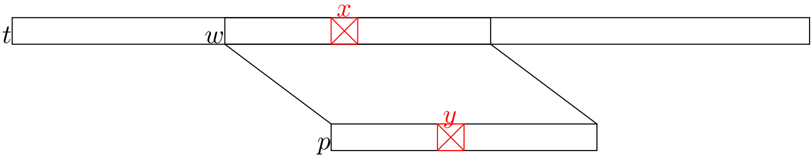

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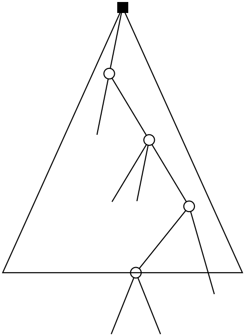

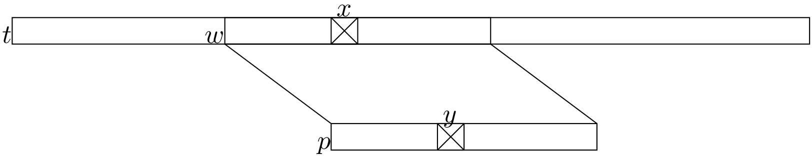

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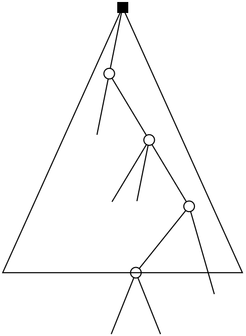

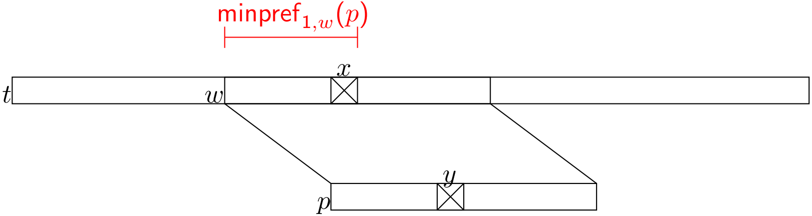

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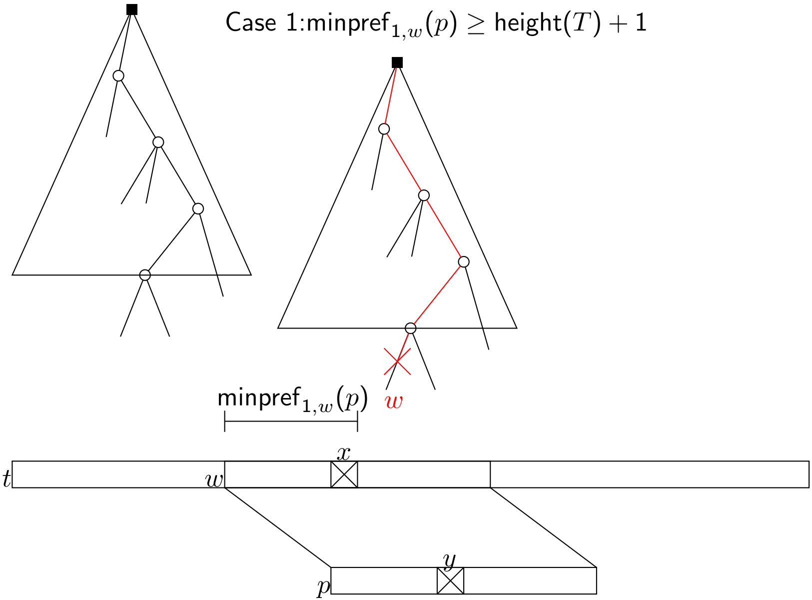

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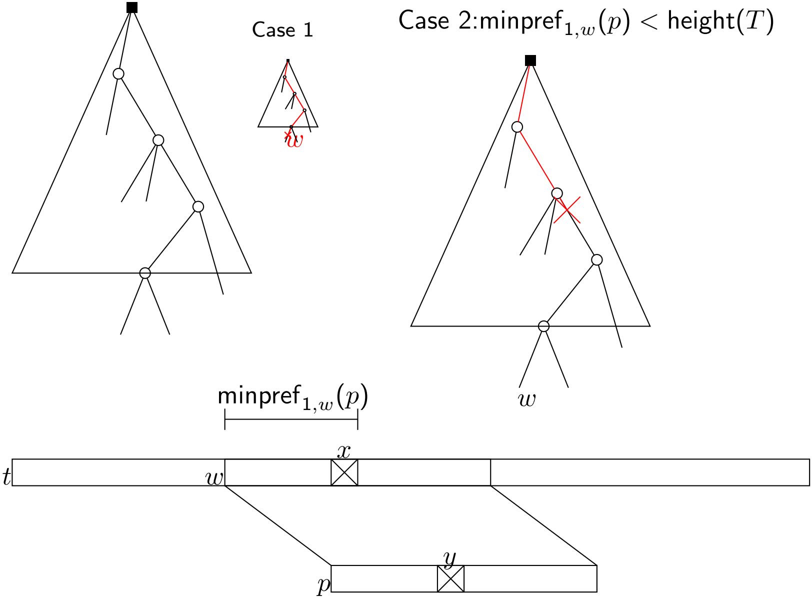

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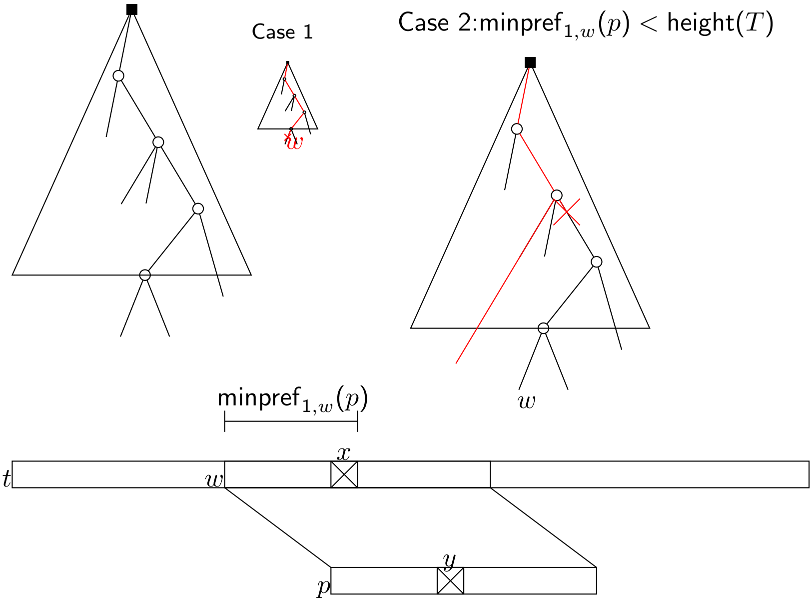

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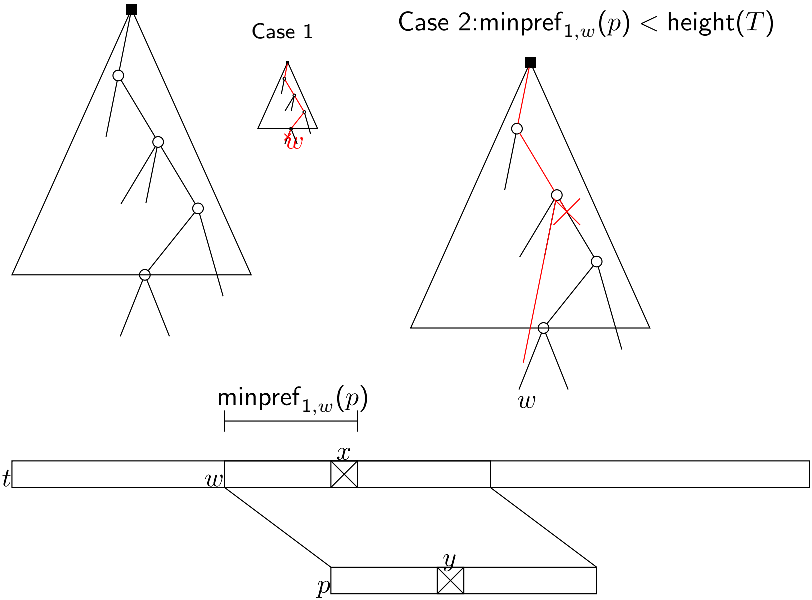

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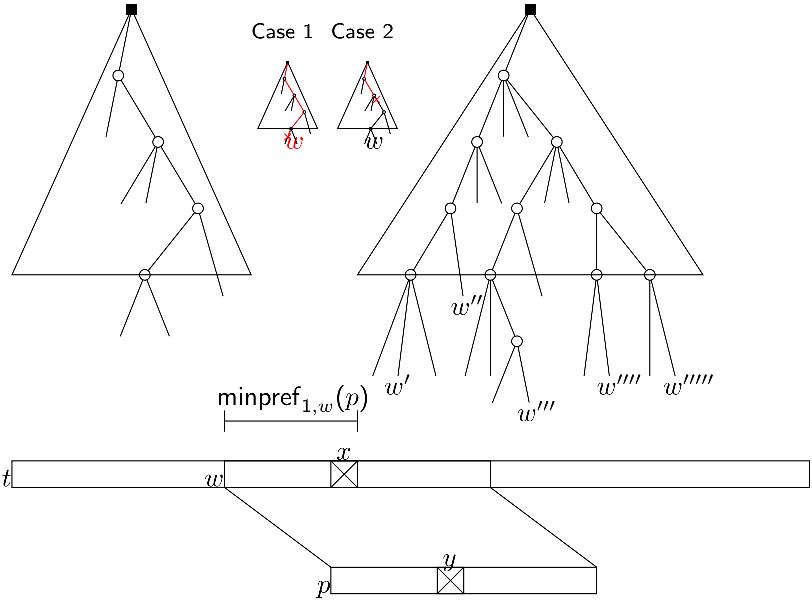

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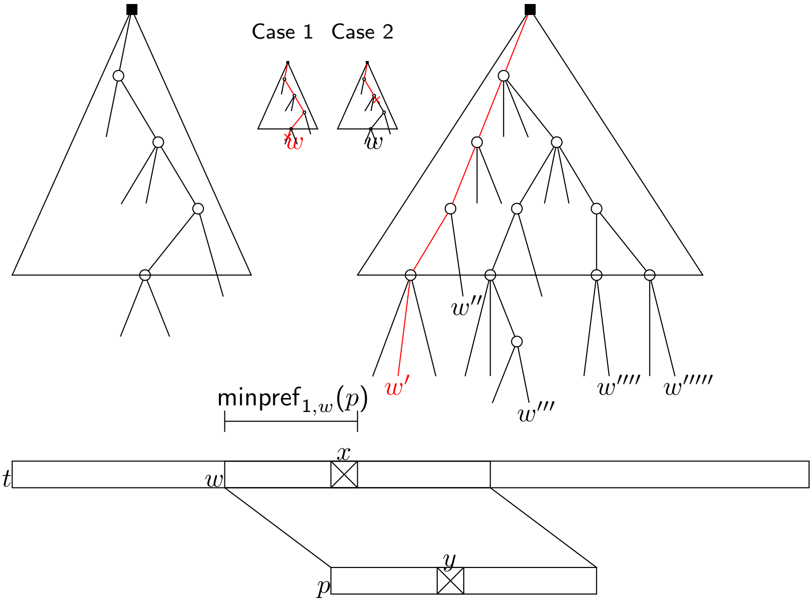

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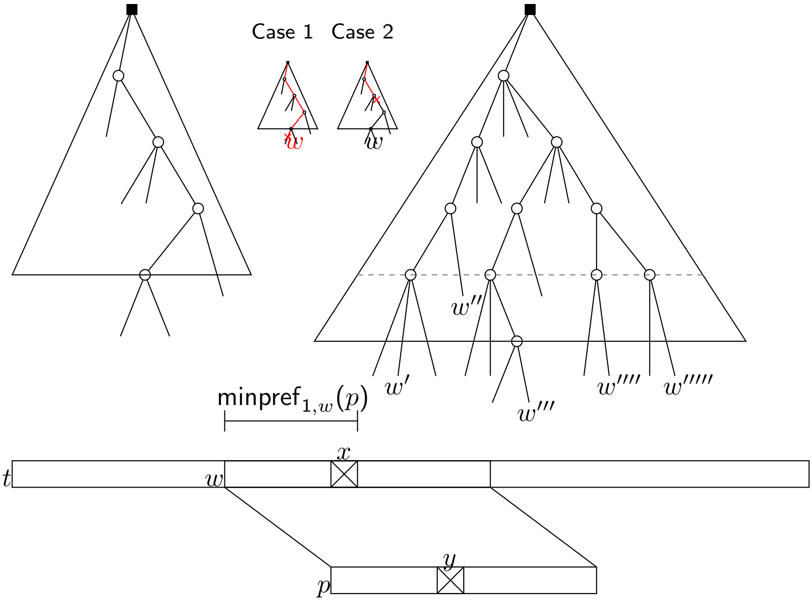

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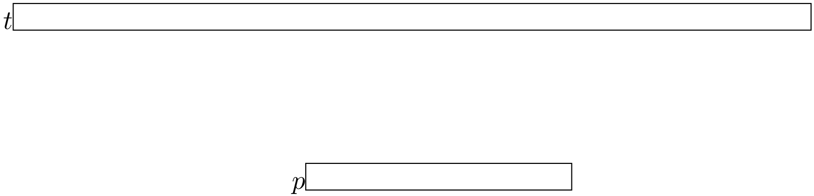

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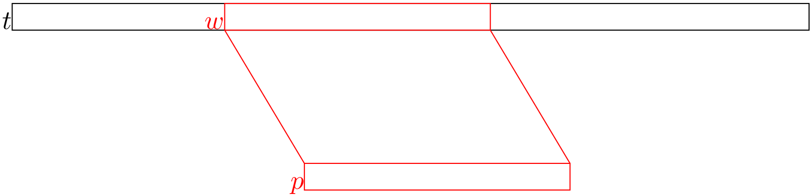

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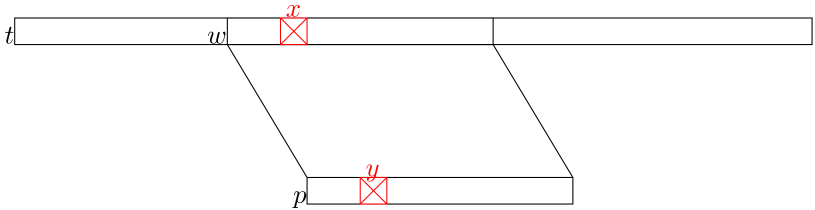

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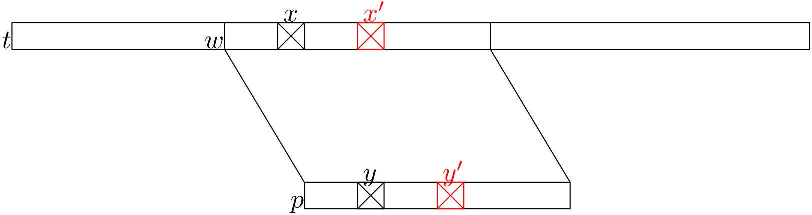

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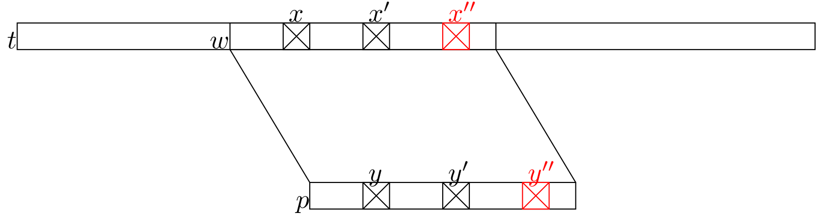

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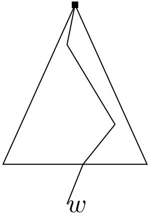

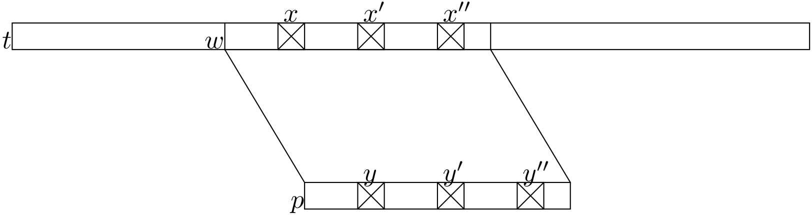

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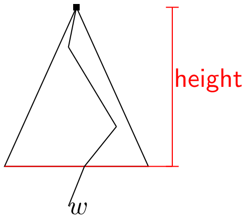

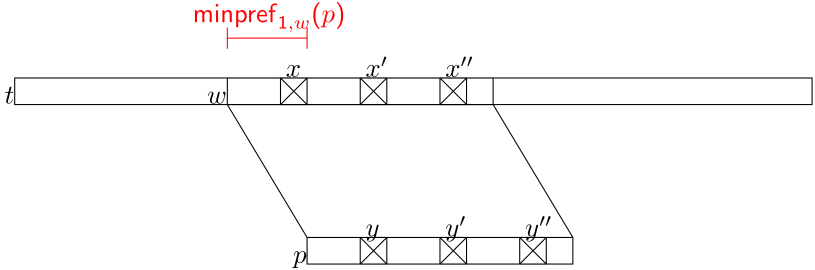

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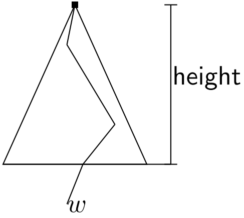

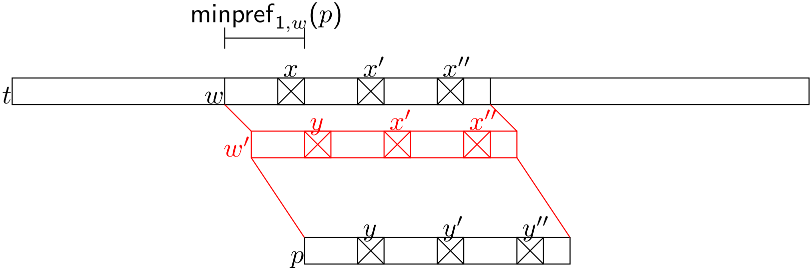

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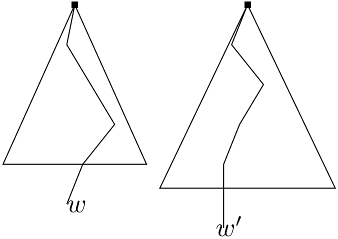

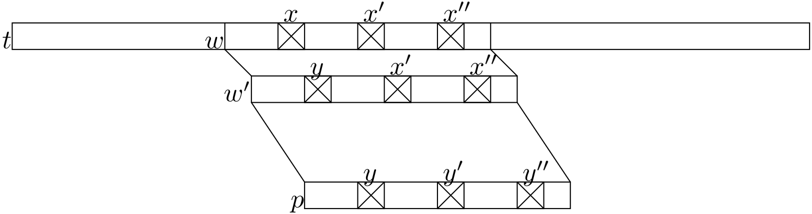

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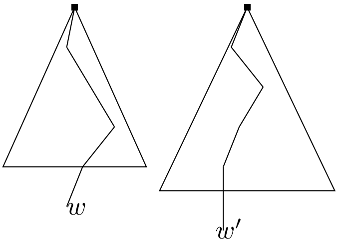

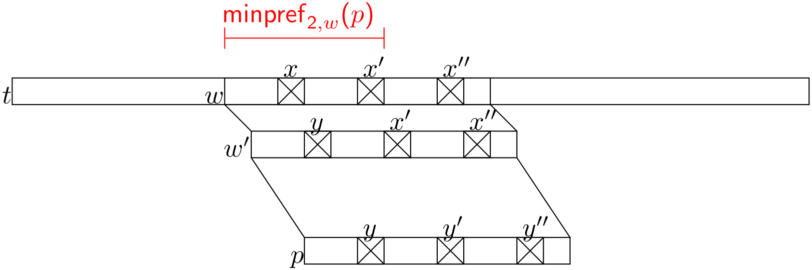

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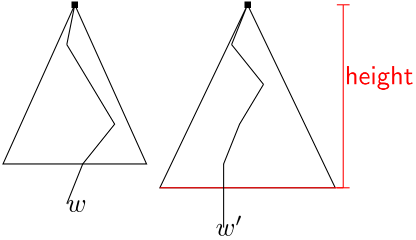

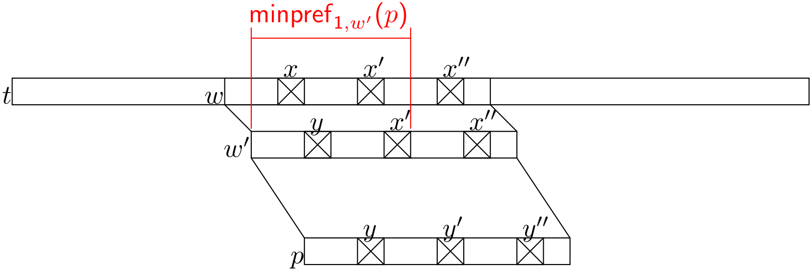

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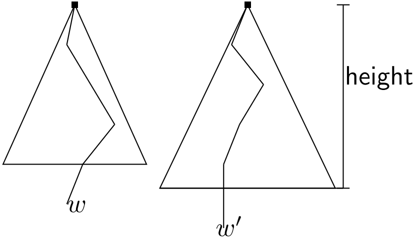

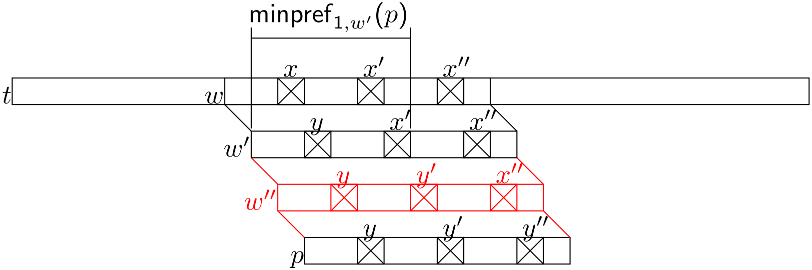

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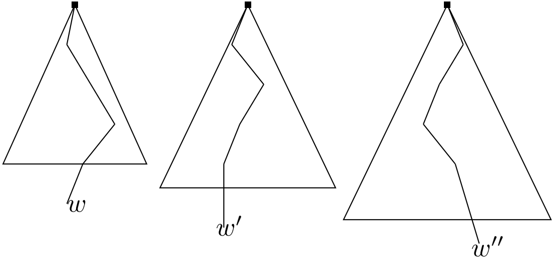

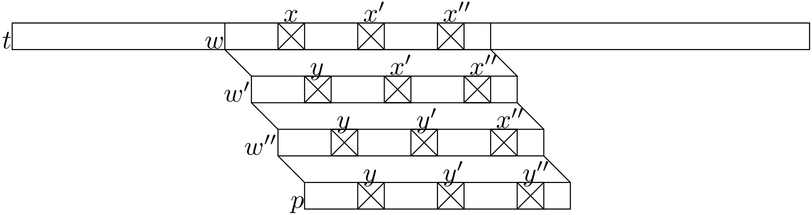

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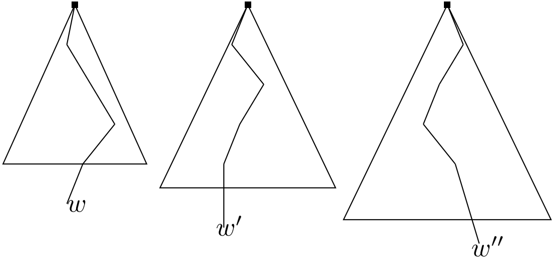

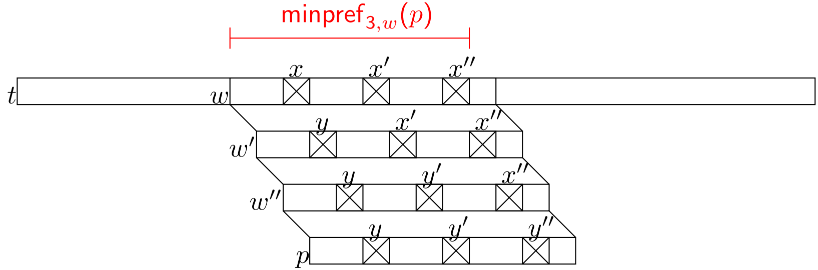

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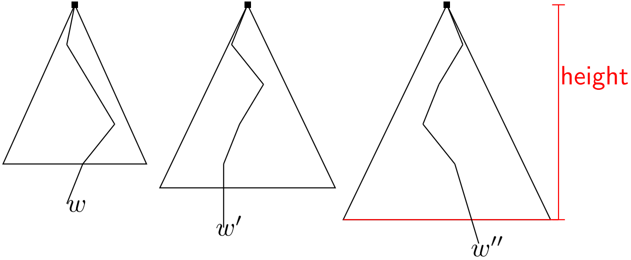

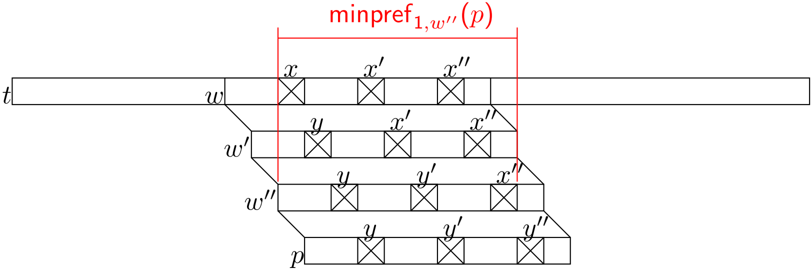

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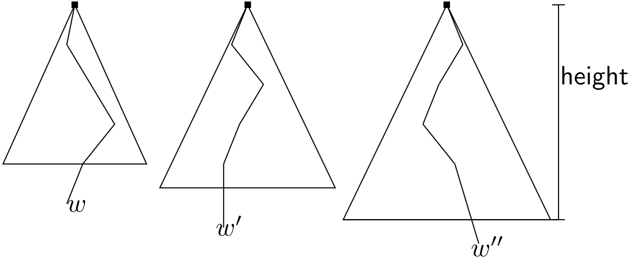

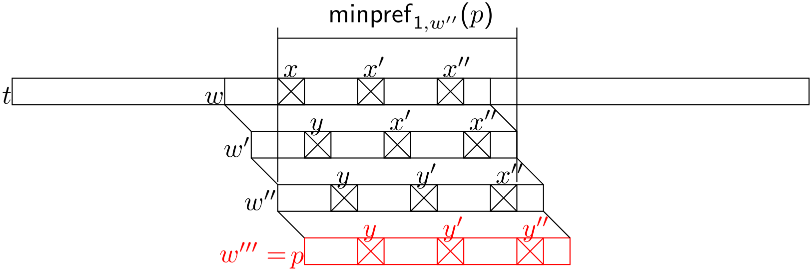

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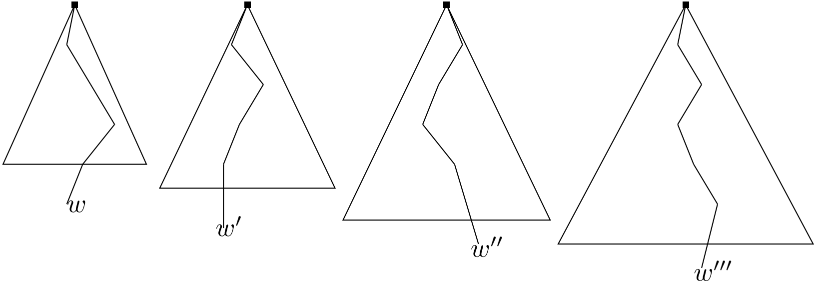

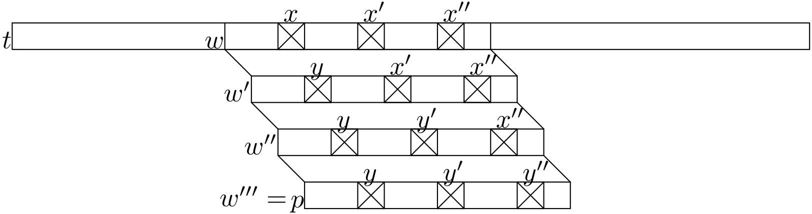

Leaves of error trees corresponding to matches

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1. reach a leaf-edge
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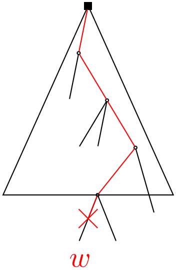
Case 1



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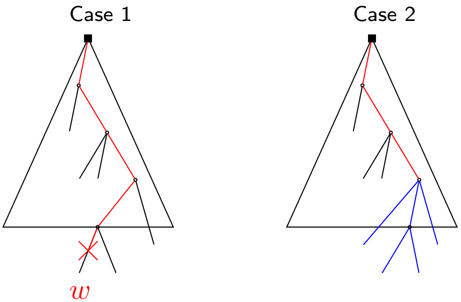

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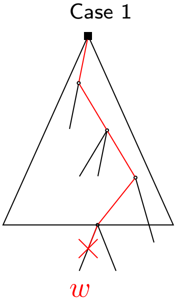

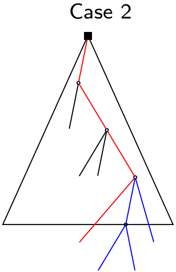

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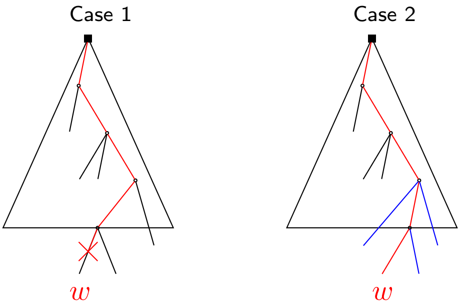

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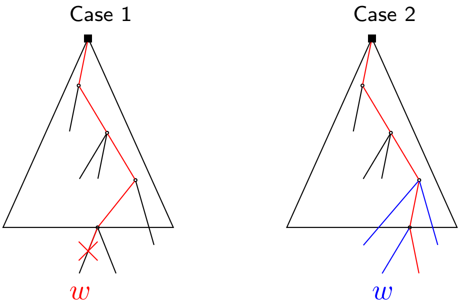

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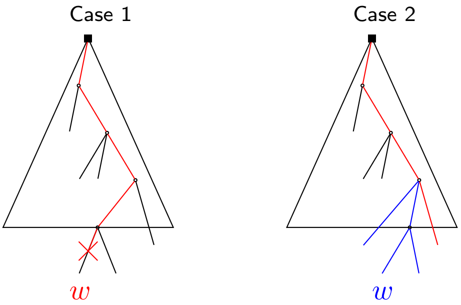
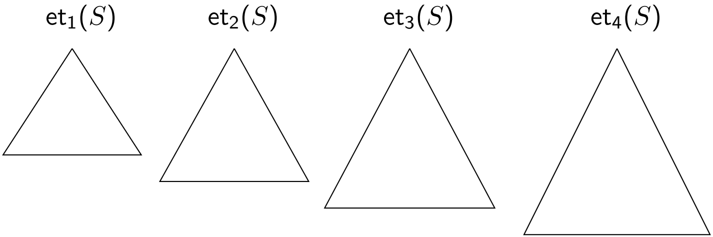


Search Time



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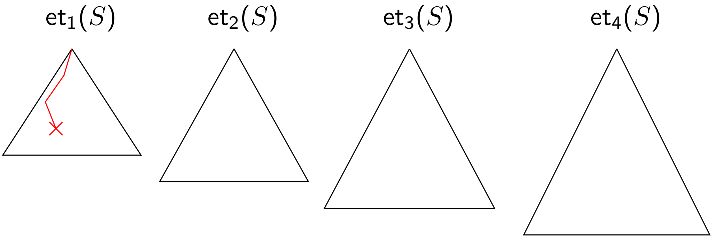

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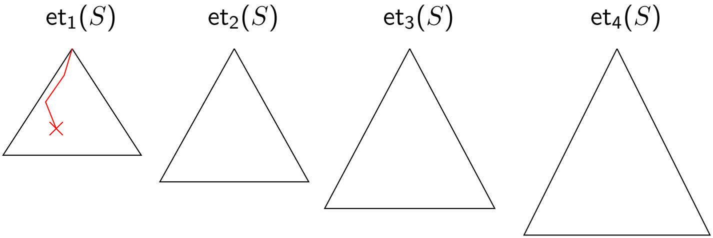
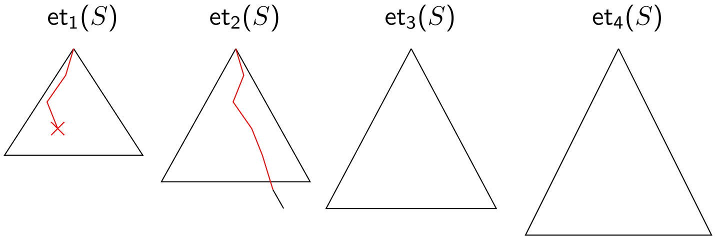


$O(m)$

Search Time



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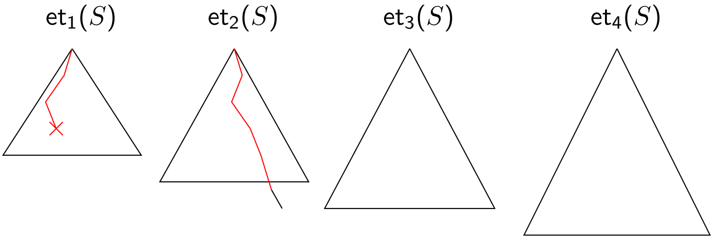

$O(m)$ $O(m + km)$

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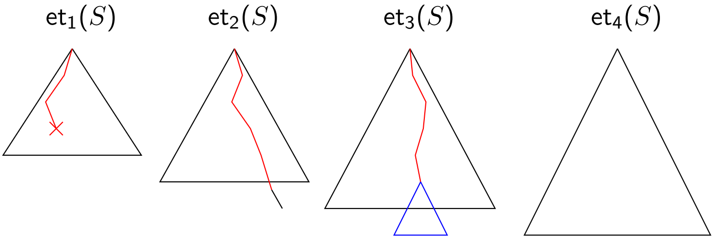

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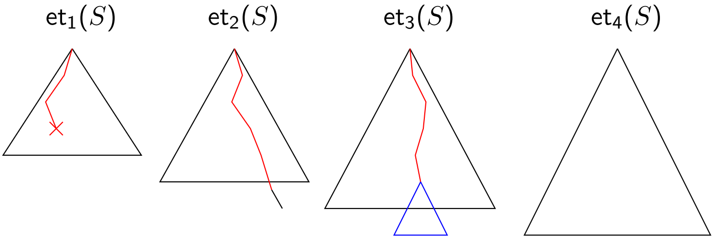

Range queries!

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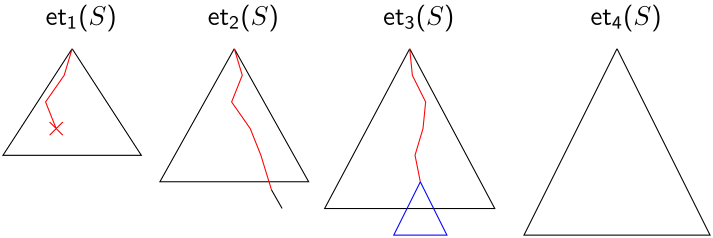

Range queries!

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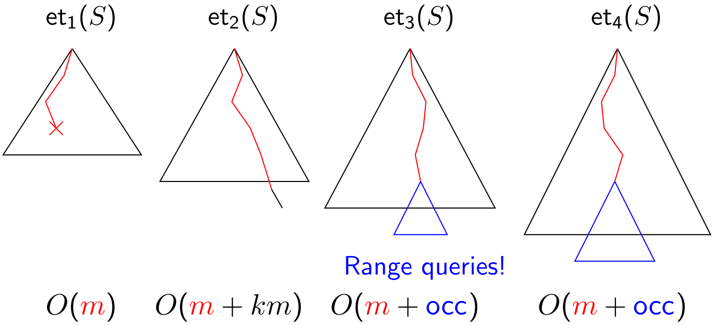
Search Time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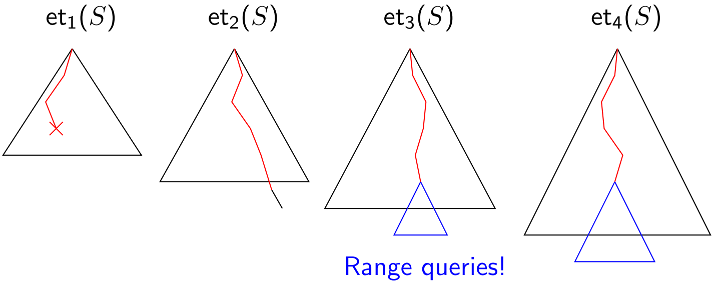

$O(m)$ $O(m + km)$ $O(m + \mathrm{occ})$ $O(m + \mathrm{occ})$

$$\Rightarrow O(m + \mathrm{occ})$$

Range Queries

Input: Array A of size n containing integer values. $< \text{Query}, \text{Preproc.} >$

RMQ: (Range Minimum Query) For (i,j) find index l with $< O(1), O(n) >$

$$A[l] = \min_{i \le k \le j} \{A[k]\}.$$

BVRQ: (Bounded Value Range Query) For (i,j,k) find all indexes $< O(|L|), O(n) >$

$$L = \{l \mid i \le l \le j \text{ and } A[l] \le k\}.$$

CRQ: (Colored Range Query) For (i,j) find the distinct set of numbers $< O(|C|), O(n) >$

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0	1	2	3	4	5	6	7	8	9
0	1	7	4	2	9	8	5	3	6

RMQ

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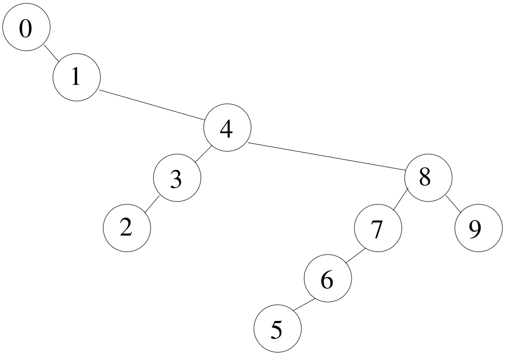

RMQ
=
Cartesian Tree
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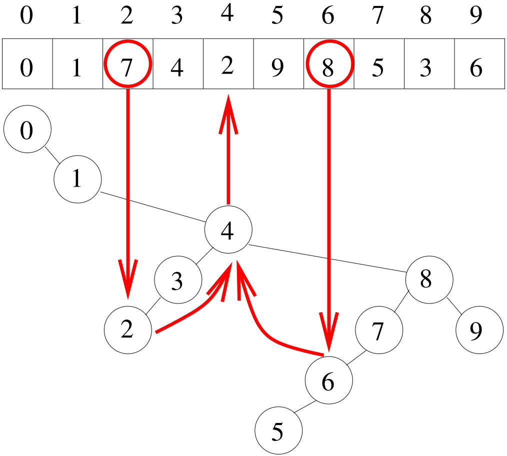
Range Minimum Queries



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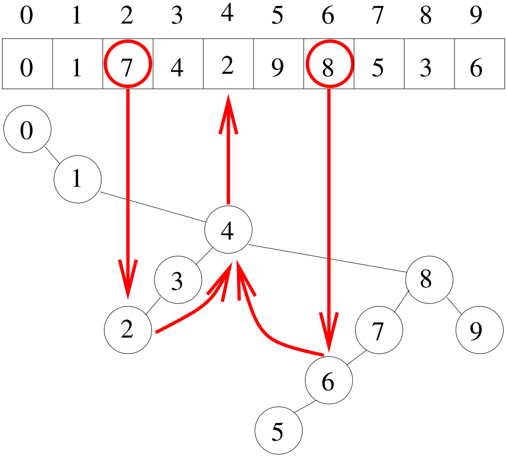

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BVRG

Bounded value range queries are solved through successive RMQs, e.g., querying $(2, 7, 6)$:

The querytree is binary and has $|L|$ inner nodes, hence we need $O(|L|)$ RMQs in total.

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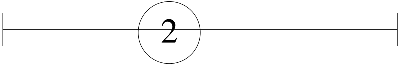

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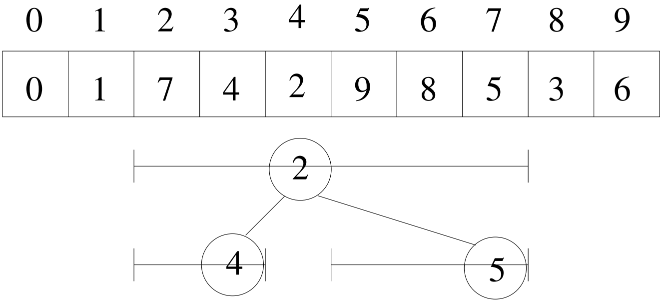

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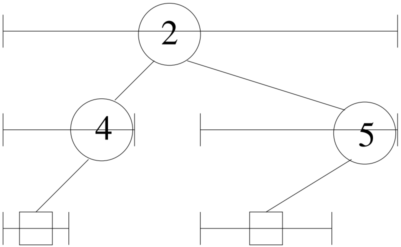

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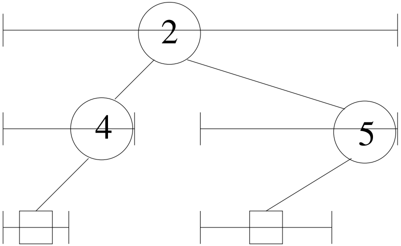

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CRQ

Colored range queries are reduced by using another array which stores the last index of each value. A CRQ (i, j) then transforms into a BVRQ $(i, j, i-1)$ on the new array:

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1	1	2	1	1	2	1	3	2	3

−1	0	−1	1	3	2	4	−1	5	7

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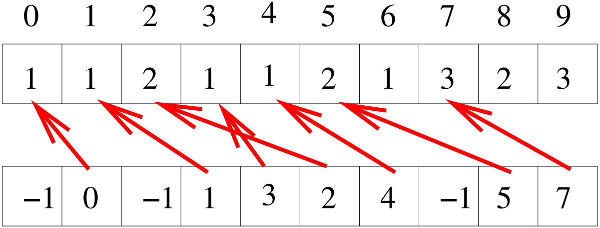

Error Sets

Definition (Error Sets)

Defined inductively by $W_0 = S$ and

$$W_i = \Gamma_{h_{i-1}}(W_{i-1}) \cap S_i \quad ,$$

where

- $\Gamma_l = \{op(u)v \mid uv \in A, |op(u)| \le l + 1, op \in \{\mathrm{del}, \mathrm{ins}, \mathrm{sub}\}\}$,
- $S_i = \{r \mid \text{there exists } s \in S \text{ such that } d(s, r) = i\}$,
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- Label leaf x by suffix number and $l = \text{minpref}_{k,s}(\text{path}(x))$.
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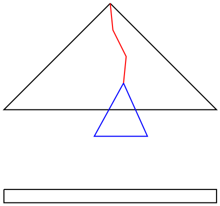
Result Selection using Range Queries

- Array of leaves A, additional array B.
- Subtree ranges at nodes.
- B contains minimal prefix l $\Rightarrow$ occurrences
- B contains starting positions $\Rightarrow$ positions
- B contains string identifiers $\Rightarrow$ documents



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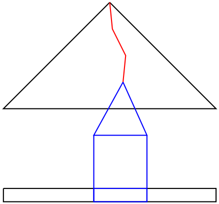

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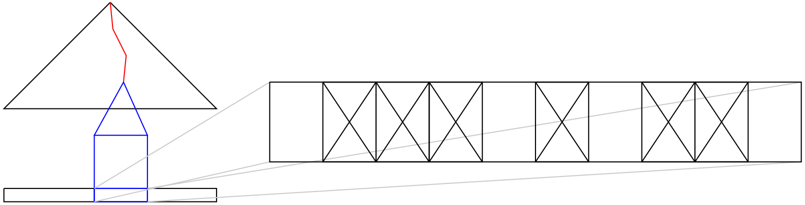

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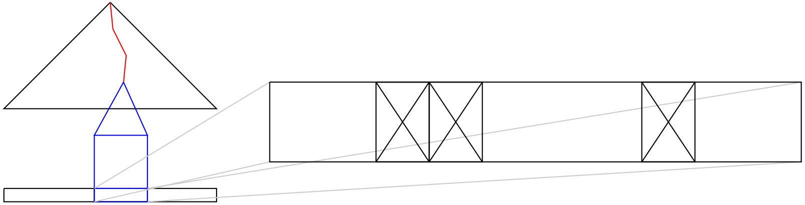

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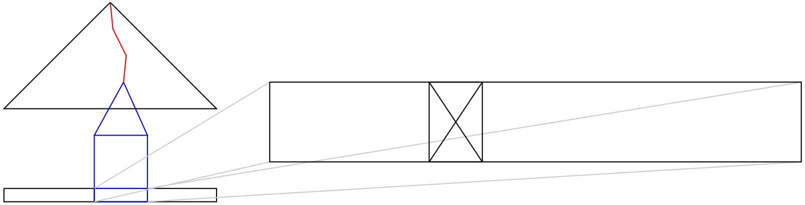

Basic Bounds

- Set $h_i = \text{maxpref}(W_i)$ (the height of $\text{et}_i(S)$)
- The error sets have size $|W_i| = O(h_0 h_1 \cdots h_{i-1} |S|)$
- Construction time for $\text{et}_i(S)$ is $O(h_0 h_1 \cdots h_i |S|)$
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Random Model

- Stationary ergodic source
- *mixing* condition

$$c_1 \Pr\{\mathcal{A}\} \Pr\{\mathcal{B}\} \leq \Pr\{\mathcal{A} \cap \mathcal{B}\} \leq c_2 \Pr\{\mathcal{A}\} \Pr\{\mathcal{B}\}.$$

- Probability of common substring:

$$r_2 = \lim_{n \to \infty} \frac{-\ln\left(\sum_{w \in \Sigma^n} (\Pr\{w\})^2\right)}{2n}.$$

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$$r_2 = \lim_{n\to\infty} \frac{-\ln\left(\sum_{w\in\Sigma^n} (\mathsf{Pr}\{w\})^2\right)}{2n}.$$

Random Model

- Stationary ergodic source
- *mixing* condition

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Height of Error Trees

- Let l be the expected length of a common substring.
- Each step adds one error to the strings.
- Possibly more matching characters afterwards, expected increment is at most l.
- $\Rightarrow h_i \leq (2i+1)l$.
- Thus,

$$\Pr\{l_{max} > (2k+1)(c''h+1)\} \leq c'k^2h^2 e^{-2(1+\epsilon)h}$$

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Size and Preprocessing

- the expected height of the first tree is $l = O(\log n)$.
- Conditioning on $l \leq c \log n$ yields the expected size

$$\begin{aligned}
\mathbf{E}[n \cdot h^k] &\leq n(ck \ln n)^k + \sum_{i=ck \ln n}^{\infty} n i^k \Pr\{l_{max} > i\} \\
&< n(ck \ln n)^k + O(1) \\
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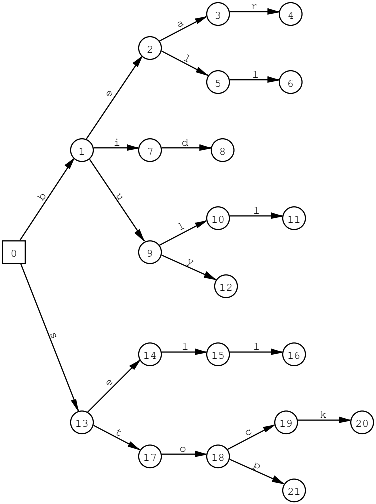
Weak Tries

Definition (Weak Trie)

For $l > 0$, the l-weak trie for a set of strings $S \subset \Sigma^*$ is a rooted tree with edges labeled by characters from Σ. For any node with a depth less than l, all outgoing edges are labeled by different characters, and there are no branching nodes with a depth of more than l. Each path from the root to a leaf can be read as a string from S.

Tries and Weak Tries



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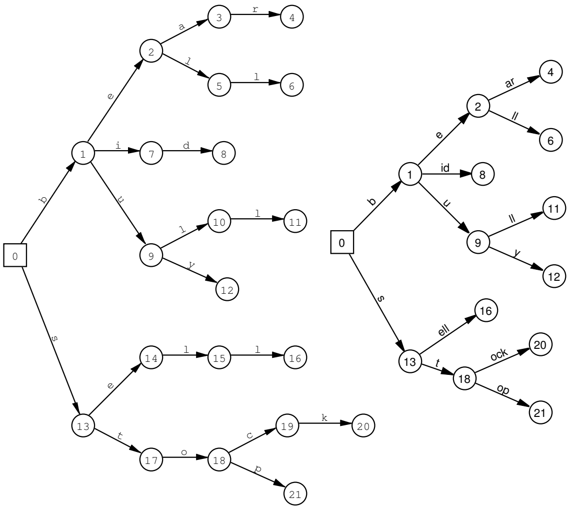

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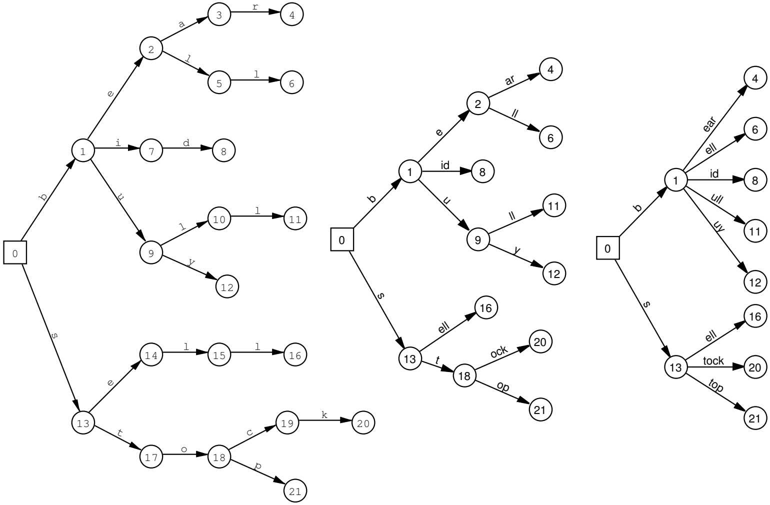

Error Tree

- Define the error trie as h_i-weak trie:

$$\mathsf{et}_i(S) = \mathcal{W}_{h_i}(W_i)$$

- Leaf labels are the same as before: leaf x is labeled by suffix number and $l = \mathsf{minpref}_{k,s}(\mathsf{path}(x))$.
- Leaf is a hit if pattern length greater than l.
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Basic Bounds

Data structure:

- Set $h_i = ck \log n + i$.
- The size is $O(n \log^k n)$.
- The preprocessing takes $O(n \log^{k+1} n)$ (weak tries!).

Searching:

- For $m \leq ck \log n$: as before in $O(m + \text{occ})$ time.
- For $m > ck \log n$: use a generalized suffix tree (as a filter).
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- Can we close the gap?
  - Our new indexing structure works best for “small” patterns of length $O(\log n)$.
  - On average, “small” patterns are the hardest to find.
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Thank you!

Questions?

Text Indexing with Errors

Moritz G. Maaß and Johannes Nowak

{maass,nowakj}@in.tum.de
Institut für Informatik
Technische Universität München

June 20, 2005

Example

```
GACTCAAAACGGGTTGTTACCGGGTATGGCTAGAATCATC
CGTACTGCGTGACCGACGGATGACGAATAAAGGAGTTAAC
TTGAGGGCGGCGAGCGACCTACAAACATGTTCGGGAAACT
CTGTCCAAAATAAACCTGAGACCAACCGTTTAGCAAGAAG
```

Pattern : ACAAC

Example - Occurrences

GACTCAAAACGGGTTGTTACCGGGTATGGCTAGAATCATC : (6,9)
CGTACTGCGTGACCGACGGATGACGAATAAAGGAGTTAAC
TTGAGGGCGGCGAGCGACCTACAAACATGTTCGGGAAACT : (20,23),(20,24),(22,25),(35,38)
CTGTCCAAAATAAACCTGAGACCAACCGTTTAGCAAGAAG : (11,14),(22,25)

Pattern : ACAAC

Example - Positions

GACTCAAAACGGGTTGTTACCGGGTATGGCTAGAATCATC : 6
CGTACTGCGTGACCGACGGATGACGAATAAAGGAGTTAAC
TTGAGGGCGGCGAGCGACCTACAAACATGTTCGGGAAACT : 20,22,35
CTGTCCAAAATAAACCTGAGACCAACCGTTTAGCAAGAAG : 11,22

Pattern : ACAAC

Example - Documents

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CGTACTGCGTGACCGACGGATGACGAATAAAGGAGTTAAC
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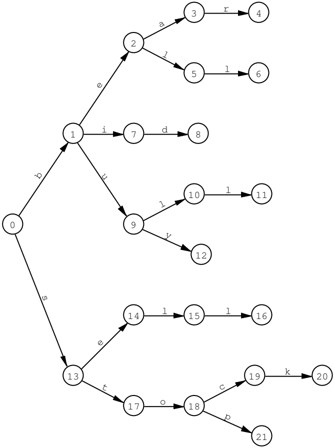
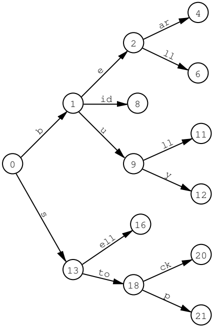
Tries and PATRICIA Trees

String set:

```
bear
bell
bid
bull
buy
sell
stock
stop
```





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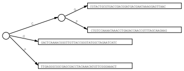

AACTCAAAACGGGTTGTTACCGGGTATGGCTAGAATCATC
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GTCTCAAAACGGGTTGTTACCGGGTATGGCTAGAATCATC
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GGTACTGCGTGACCGACGGATGACGAATAAAGGAGTTAAC
TGTACTGCGTGACCGACGGATGACGAATAAAGGAGTTAAC
CATACTGCGTGACCGACGGATGACGAATAAAGGAGTTAAC
CCTACTGCGTGACCGACGGATGACGAATAAAGGAGTTAAC
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GTGAGGGCGGCGAGCGACCTACAAACATGTTCGGGAAACT
TAGAGGGCGGCGAGCGACCTACAAACATGTTCGGGAAACT
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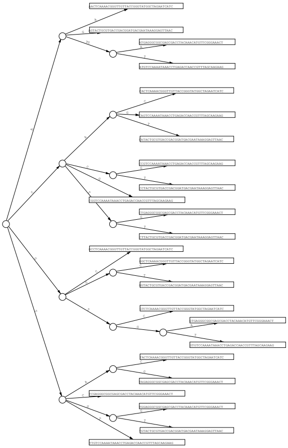

Edit Distance

- Dynamic programming version:

$$D_{i,j} = \min\{D_{i-1,j} + 1, D_{i-1,j-1} + \delta(i,j), D_{i,j-1} + 1\},$$

for all $1 \leq i \leq |u|$, $1 \leq j \leq |v|$.

$$\delta(i,j) = \begin{cases} 1 & \text{if } u[i] \neq v[j] \\ 0 & \text{if } u[i] = v[j] \end{cases}$$

- As operator $op : \Sigma^* \to \Sigma^*$, i.e.,

$$\begin{aligned} op_{del,2}(\mathtt{ACAAC}) &= \mathtt{AAAC} \\ op_{sub,2,\mathtt{A}}(\mathtt{ACAAC}) &= \mathtt{ACAAA} \end{aligned}$$